The Algorithm of Frolov for numerical integration

Master Thesis
to obtain the academic degree of
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in the Master's Program
Mathematik in den Naturwissenschaften
Statutory Declaration

I hereby declare that the thesis submitted is my own unaided work, that I have not used other than the sources indicated, and that all direct and indirect sources are acknowledged as references.
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1 Introduction

In numerical integration and approximation theory we seek to find good methods to approximate the functional

\[ I(f) = \int_{[0,1]^d} f(x)dx. \]

The goal hereby is a small approximation error given certain resource bounds. Many \(d\)-dimensional integrals can be transformed to an integral of this type. Therefore studying the integral with the unit cube as integration domain is indeed meaningful.

We study functions \(f\) of Sobolev spaces \(W^{k,q}\) and try to approximate \(I(f)\) for them. The space \(W^{k,q}\) contains functions with existing weak derivatives up to order \(k\) in each component, which are \(L^q\)-integrable.

As usual in numerical integration theory the integral gets approximated by a finite sum. Therefore we use quadrature formulas, i.e. expressions of the form

\[ Q(f) = \sum_{j=1}^{N} a_j \cdot f(x_j) \]

with nodes \((x_j)_{j=1}^N, \ x_j \in [0,1]^d\), and weights \((a_j)_{j=1}^N\) with \(a_j \in \mathbb{R}\). The nodes and weights should be chosen such that \(Q(f) \approx I(f)\) holds.

We define the worst-case error \(e(Q,H)\) for a given quadrature formula \(Q\) applied on elements of a set \(H\) of functions by

\[ e(Q, H) = \sup_{f \in H, \|f\| \leq 1} |I(f) - Q(f)|. \]

If \(N\) is the number of sample points \((x_j)\) of \(Q\), we get the optimal order of convergence \(O\left( \frac{(\log N)^{(d-1)/2}}{N^k} \right)\) for the worst-case error of functions of Sobolev spaces \(W^{k,q}\).

Beside Sobolev spaces we also study functions of bounded variation. For the error estimation, Frolov used lattices constructed by specially chosen Vandermonde matrices. In this thesis we will take a closer look at this construction.

The introductory paper written by Mario Ullrich [17] related to the paper of Frolov and the paper about construction of uniform distributions form Skriganov [13] were used as basis for this thesis.
2 Lattice theory

First of all we need to introduce the theory of lattices. We state some basic properties and concepts. Later we will choose the nodes from our quadrature formula as lattice points and, therefore, we need these properties of lattices to find upper error bounds.

2.1 Lattice generator

Definition 2.1. Let \( g_1, ..., g_n \in \mathbb{R}^d \) be linearly independent vectors and \( G = (g_1 \mid ... \mid g_n) \in \mathbb{R}^{d \times n} \). Then we denote the set of all linear combinations of \( g_1, ..., g_n \) with integer coefficients by

\[
L = G(\mathbb{Z}^d) = \left\{ \sum_{i=1}^{n} m_i \cdot g_i : m_1, ..., m_n \in \mathbb{Z} \right\}
\]

as the lattice \( L \) of rank \( n \) generated by \( G \).

Further we call the vectors \( g_1, ..., g_n \) a basis and the matrix \( G \) a generator of the lattice \( L \).

If we combine the definition of a lattice generated by a matrix \( G \) with the definition of the vector space generated by \( G \), i.e. \( \text{span}(G) = \{ \sum_{i=1}^{n} x_i \cdot g_i : x_1, ..., x_n \in \mathbb{R} \} \), then we see that the only difference lies in the value range of the coefficients.

This leads us to the fact that every basis \( g_1, ..., g_n \) of a lattice \( L \) is also a basis of \( \text{span}(G) \), but not every basis of \( \text{span}(G) \) is a basis of \( L \). As a counterexample we can simply use \( 2 \cdot g_1, ..., 2 \cdot g_n \). Obviously \( g_1 \in \text{span}(2 \cdot g_1, ..., 2 \cdot g_n) \), but \( g_1 \notin 2G(\mathbb{Z}^d) \).

Remark 2.2. To use just integer coefficients in the definition of a lattice also implies that a lattice \( L \subset \mathbb{R}^d \) is a discrete subset\(^1 \) of \( \mathbb{R}^d \). By definition \( L \) is closed under addition, contains inverse elements with respect to the addition and the zero-vector. Hence, \((L; +)\) is an additive, discrete subgroup of \((\mathbb{R}^d; +)\).

Definition 2.3. Let \( L \) be a lattice of \( \mathbb{R}^d \). We call \( L \) a full rank lattice, if the rank of \( L \) is \( d \).

The simplest example of a lattice is \( \mathbb{Z}^d \). The generator of \( \mathbb{Z}^d \) is the \( d \)-dimensional identity matrix and \( \mathbb{Z}^d \) is a full rank lattice.

For sets \( A, B \subseteq \mathbb{R}^d \) and \( t \in \mathbb{R} \), \( T = (t_1, ..., t_d) \in \mathbb{R}^d \) we define the operations

\[
A + B = \{ a + b : a \in A, b \in B \}, \quad t \cdot A = \{ t \cdot a : a \in A \}
\]

and

\[
T \cdot A = \{(t_1 \cdot a_1, ..., t_d \cdot a_d) : a = (a_1, ..., a_d) \in A\}.
\]

Further we use the notation

\[
A_{T, B} = T \cdot A + B.
\]

\(^1\)A set \( L \subset \mathbb{R}^d \) is called a discrete subset of \( \mathbb{R}^d \), if for each point of \( L \) there exists an open neighbourhood which does not contain any other point of \( L \).
If $B$ is a single point $b \in \mathbb{R}^d$ we still write $A_{T,b}$.

Let $L \subset \mathbb{R}^d$ be a lattice and $t \in \mathbb{R}$. Then the set $t \cdot L = \{ t \cdot \ell : \ell \in L \}$ is again a lattice. Similarly, for a given vector $T = (t_1, ..., t_d) \in \mathbb{R}^d$ the set $T \cdot L$ is a lattice. This case coincides with the case above if $T = (t, ..., t)$ for a $t \in \mathbb{R}$.

In this thesis we study sets of the type

$$T \cdot L + x$$

with $x \in \mathbb{R}^d$. Similarly as above we refer to this set by using the abbreviation $L_{T,x}$.

This set can be seen as a shifted lattice, which is dilated differently in every direction. It is still a discrete set, but not necessarily a lattice, since in general $0 \notin L_{T,x}$ and zero is an element of each lattice.

**Definition 2.4.** Let $L$ be a lattice of rank $n$ with basis vectors $g_1, ..., g_n$. We call the parallelepiped

$$\mathcal{P}(G) = \left\{ \sum_{i=1}^{n} x_i \cdot g_i : x_1, ..., x_n \in [0,1) \right\}$$

generated by the basis vectors the *fundamental cell* of $L$, sometimes also the *fundamental region*.

Since a fundamental cell $\mathcal{P}(G)$ of a lattice $L$ with generator $G$ is half-open, the set of all translations

$$\mathcal{P}(G) + \ell = \{ x + \ell : x \in \mathcal{P}(G) \}$$

by lattice points $\ell \in L$ forms a partition of $\text{span}(G)$.

We have to emphasize that $\mathcal{P}(G)$ depends on the chosen basis vectors. The following picture shows two different possibilities for the same lattice.

![Figure 1: Two different choices of basis vectors for the same lattice resulting in two different fundamental cells.](image-url)
However, not every choice of linearly independent vectors gives a basis of the lattice as we have already seen. The following lemma states exactly which possibilities do exist.

**Lemma 2.5.** Let $L$ be a lattice of rank $n$ and $g_1, \ldots, g_n \in L$ be linearly independent. Then $g_1, \ldots, g_n$ form a basis of $L$ if and only if $\mathcal{P}(G) \cap L = \{0\}$.

**Proof.** First we assume that $g_1, \ldots, g_n$ form a basis of $L$. $L$ is defined as the set of linear combination of $g_1, \ldots, g_n$ with integer coefficients, whereas $\mathcal{P}(G)$ contains the linear combinations of $g_1, \ldots, g_n$ with coefficients in $[0,1)$. Therefore the zero vector is the only point in $\mathcal{P}(T) \cap L$.

Now let us assume that $g_1, \ldots, g_n$ are linearly independent and $\mathcal{P}(G) \cap L = \{0\}$. Since $L$ has rank $n$ we can express every point $x \in L$ as

$$x = \sum_{i=1}^{n} y_i \cdot g_i$$

for some $y_1, \ldots, y_n \in \mathbb{R}$. Due to the closedness of $L$ under addition we know that

$$x' = \sum_{i=1}^{n} (y_i - \lfloor y_i \rfloor) \cdot g_i \in L$$

as well. By our assumption, and since $y_i - \lfloor y_i \rfloor \in [0,1)$ for all $i \in \{1, \ldots, n\}$, we have that $x' = 0$. Hence, all the coefficients $y_i - \lfloor y_i \rfloor$ are integers, which is only possible if all $y_1, \ldots, y_n \in \mathbb{Z}$. Therefore $x$ is a linear combination of $g_1, \ldots, g_n$ with just integer coefficients.

**Definition 2.6.** A matrix $U \in \mathbb{Z}^{n \times n}$ is called unimodular if $|\det U| = 1$. 

![Figure 2: The fundamental cell $\mathcal{P}((g_1 \mid g_2))$.](image-url)
Let $U \in \mathbb{Z}^{n \times n}$ be a unimodular matrix. Then $U^{-1}$ exists and has determinant

$$|\det (U^{-1})| = \frac{1}{|\det U|} = 1.$$ 

Further

$$U^{-1} = \frac{1}{\det U} \cdot \text{Cof}(U)^T = \pm \text{Cof}(U)^T,$$

where $\text{Cof}(U) = (c_{ij})_{ij}$ is the cofactor matrix of $U$ with entries $c_{ij} = (-1)^{i+j} \cdot M_{ij}$ and $M_{ij}$ are the minors of $U$, i.e. the determinant of $U$ with deleted $i$th row and $j$th column. Therefore $U^{-1}$ has just integer entries and is unimodular as well.

With the following lemma we can also predict if two given bases form the same lattice or not.

**Lemma 2.7.** Let $G$ be a generator of a lattice $L_1$ and let $H$ be a generator of a lattice $L_2$. Then $L_1 = L_2$ if and only if there exists a unimodular matrix $U$ such that $G = HU$.

**Proof.** First we assume that $L_1 = L_2$. Since all columns $g_1, ..., g_n$ of $G$ are elements of $L_2$ generated by $H$, by definition of a lattice we have that there exists an integer matrix $U$ such that $G = HU$. Similarly, there exists a matrix $V \in \mathbb{Z}^{n \times n}$ such that $H = GV$. Hence, we have that $H = GV = HUV$ and, therefore,

$$H^T H = (HUV)^T (HUV) = (UV)^T H^T H (UV).$$

Since the determinant is multiplicative, we have

$$\det (H^T H) = \det (UV)^2 \cdot \det (H^T H).$$

By definition of a generator we know that $\det (H^T H) \neq 0$ and therefore

$$\det (UV)^2 = 1.$$ 

Thus,

$$|\det U \cdot \det V| = 1.$$ 

Since both $U$ and $V$ have just integer entries and the determinant is a linear combination of the entries, we obtain

$$|\det U| = 1 = |\det V|.$$ 

Hence, $U$, and also $V$, is a unimodular matrix.

Now let us assume that $G = HU$ for a unimodular matrix $U$. Then each column of $G$ is an element of $L_2$ and we have that $L_2 \subseteq L_1$. Since $U^{-1}$ is unimodular we also have that $H = GU^{-1}$ and, by the same argument as before, $L_1 \subseteq L_2$. Hence, we conclude that $L_1 = L_2$.

**Definition 2.8.** Let $L \subset \mathbb{R}^d$ be a lattice with a generator $G$ of rank $n$. Then we define the determinant of the lattice as the $n$-dimensional volume of the fundamental cell

$$\det L = \text{vol}_n(\mathcal{P}(G)).$$
Remark 2.9. Let $L$ be a lattice with a generator $G$. Since $\mathcal{P}(G)$ is a parallelootope, the volume of the fundamental cell can also be expressed as

$$\det L = \sqrt{\det (G^T G)}.$$ 

The determinant $\det (G^T G)$ gets often referred to as the Gram determinant.

In particular, if a lattice $L$ has full rank and $G$ is a generator of $L$, it follows that

$$\det L = |\det G|.$$ 

Further there always exists a generator $G$ such that $\det G = \det L$.

Corollary 2.10. Let $L$ be a lattice. Then the determinant of the lattice $\det L$ is independent of the chosen lattice basis.

Proof. Let $G$ and $H$ be generators of $L$. Due to Lemma 2.7 we know that there exists a unimodular matrix $U$ such that $G = HU$. Thus,

$$\det (G^T G) = \det ((HU)^T (HU)) = \det U \cdot \det (H^T H) \cdot \det U = \det (H^T H).$$

By Remark 2.9, this proves the corollary. □

Geometrically the determinant of a lattice $L$ represents the inverse of the density of lattice points. The number of lattice points of $L$ in the unit cube $[0,1]^d$ is approximately $\frac{1}{\det L}$.

Successive minima

We denote by $|.|$ the Euclidean norm of $\mathbb{R}^d$.

Definition 2.11. Let $L$ be a lattice of rank $n$ with a generator $G$. Then we denote for $i = 1,...,n$ by

$$\lambda_i(L) = \inf \{ r > 0 : \dim(\text{span}(\bar{B}(0,r) \cap L)) \geq i \},$$

where $\bar{B}(0,r) = \{ x \in \mathbb{R}^d : |x| \leq r \}$ is the closed ball of radius $r$ around zero, the successive minima of $L$.

The successive minima $\lambda_1,...,\lambda_n$ of a lattice $L$ represent the length of the basis vectors of $L$ with the smallest Euclidean norm in ascending order. In particular,

$$\lambda_1(L) = \min_{\ell \in L \setminus \{0\}} |\ell|$$
is the length of the shortest possible basis vector.

Figure 3: The successive minima of a lattice in $\mathbb{R}^2$ with two possible basis vectors of minimal length.

**Theorem 2.12** (Minkowski’s Second theorem). Let $L$ be a lattice of rank $n$ with successive minima $\lambda_1, ..., \lambda_n$. Then the inequality

$$\frac{2^n}{n!} \cdot \det L \leq \text{vol}_n(\bar{B}(0,1)) \cdot \prod_{j=1}^{n} \lambda_j \leq 2^n \cdot \det L$$

holds.

**Proof.** See [2, Kapitel 8; Theorem 5].
2.2 Admissible lattices

**Definition 2.13.** We call a lattice $L \subset \mathbb{R}^d$ admissible if

$$\inf_{\ell \in L \setminus \{0\}} \prod_{j=1}^{d} |\ell_j| > 0$$

with $\ell = (\ell_1, ..., \ell_d)$ holds.

For an admissible lattice $L$ we know for all non-zero elements $\ell \in L$ that $\ell$ has no vanishing component. An important example of a non-admissible lattice is $\mathbb{Z}^d$.

![Figure 4: Left: An admissible lattice. Right: A non-admissible lattice.](image)

**Remark 2.14.** The admissibility of a lattice with an arbitrary generator $G$ can be written equivalently as:

There exists a $k > 0$ such that for all $m \in \mathbb{Z}^n \setminus \{0\}$

$$\prod_{j=1}^{n} |(Gm)_j| \geq k \quad (2.1)$$

holds. The supremum among those $k \in \mathbb{R}$ such that (2.1) holds, we call the **admissibility constant**. For non-admissible lattices we set the admissibility constant to 0.

Let $L \subset \mathbb{R}^d$ be an admissible lattice with admissibility constant $k$. Then for $T = (t_1, ..., t_d) \in (\mathbb{R} \setminus \{0\})^d$ we obtain

$$\inf_{\ell \in T \cdot L \setminus \{0\}} \prod_{j=1}^{d} |\ell_j| = \inf_{\ell \in L \setminus \{0\}} \prod_{j=1}^{d} |t_j \cdot \ell_j| = \prod_{j=1}^{d} |t_j| \inf_{\ell \in L \setminus \{0\}} \prod_{j=1}^{d} |\ell_j| > 0.$$ 

Therefore, the lattice $T \cdot L$ is an admissible lattice too and has admissibility constant $k \cdot \prod_{j=1}^{d} |t_j|$.
Let
\[ A(\Omega; L) := |\Omega \cap L| \]
be the number of points in \( \Omega \cap L \).

**Lemma 2.15.** Let \( L \subset \mathbb{R}^d \) be an admissible lattice with admissibility constant \( k \). Then for \( T = (t_1, \ldots, t_d) \in \mathbb{R}^d \) with no vanishing component and \( x \in \mathbb{R}^d \) the upper bound
\[ A(T \cdot [0, 1]^d + x; L) \leq \frac{\prod_{j=1}^d |t_j|}{k} + 1 \]
holds.

**Proof.** Let \( \Omega = T \cdot [0, 1]^d + x \). First we assume that \( \text{vol}_d(\Omega) = \prod_{j=1}^d |t_j| < k \). If \( \Omega \) contains two distinct points \( z, z' \in L \), \( \Omega \) covers the axis-parallel box generated by \( z \) an \( z' \). Hence, the point \( z'' = z - z' \in L \) satisfies
\[ \text{vol}_d(\Omega) \geq \prod_{j=1}^d |z_j - z'_j| = \prod_{j=1}^d |z''_j|. \]
By using (2.1) we have
\[ \prod_{j=1}^d |z''_j| \geq k \]
in contradiction to the assumption. Therefore \( \Omega \) can contain at most one point of \( L \).

Let \( \text{vol}_d(\Omega) \geq k \). Now separate \( \Omega \) along a coordinate into \( \lfloor \frac{\text{vol}_d(\Omega)}{k} + 1 \rfloor \) many pieces of the same size. They have a volume less than \( k \) and contain at most one point due the argumentation above. Altogether we have
\[ A(\Omega; L) \leq \left\lfloor \frac{\text{vol}_d(\Omega)}{k} + 1 \right\rfloor \leq \frac{\text{vol}_d(\Omega)}{k} + 1. \]

An important implication of Lemma 2.15 is the fact that all axis-parallel boxes \( \Omega = T \cdot [0, 1]^d + x \) with \( 0 \in \Omega \) and \( \text{vol}_d(\Omega) < k \) satisfy
\[ A(\Omega; L \setminus \{0\}) = 0 \]
holds. If we consider the infinite union of all axis-parallel boxes \( \Omega \) containing \( 0 \) with volume at most \( k \), we get the so-called hyperbolic cross
\[ S_d = \left\{ x \in \mathbb{R}^d : \prod_{j=1}^d |x_j| < k \right\} \]
containing no point of the lattice except the zero point, i.e. \( L \cap S_d = \{0\} \).
Since lattices are invariant in respect of translation by a lattice point, this property holds for every \( \ell \in L \).
Another direct implication of Lemma 2.15 is the following Corollary:

**Corollary 2.16.** Let $L \subset \mathbb{R}^d$ be an admissible lattice. Then for every $a \in \mathbb{R}$ the hyperplane $\{x \in \mathbb{R}^d : x_j = a\}$ with $j \in \{1, \ldots, d\}$ contains at most one point of $L$.

**Proof.** Let $k > 0$ be the admissibility constant of $L$ and $a \in \mathbb{R}$. We assume that there exist two distinct points $x_1, x_2 \in \{x \in \mathbb{R}^d : x_j = a\}$. However, there exist $T, x \in \mathbb{R}^d$ such that

$$x_1, x_2 \in T \cdot [0, 1]^d + x$$

and

$$\prod_{j=1}^{d} |t_j| < k.$$

But this contradicts to the upper bound in Lemma 2.15 and therefore our assumption must be wrong. \qed
2.3 Dual lattices

**Definition 2.17.** Let \( L \) be a full rank lattice of \( \mathbb{R}^d \) and let \( (.,.) \) be the Euclidean inner product. We denote by

\[
L^\perp = \{ z \in \mathbb{R}^d : \langle \ell, z \rangle \in \mathbb{Z} \text{ for all } \ell \in L \}
\]

the *dual lattice* corresponding to \( L \).

For \( t > 0 \) the lattice \( t \cdot \mathbb{Z}^d \) has the corresponding dual lattice \( \frac{1}{t} \cdot \mathbb{Z}^d \). So a generator of the dual lattice of \( t \cdot \mathbb{Z}^d \) is the \( d \)-dimensional diagonal matrix with diagonal entries equal to \( \frac{1}{t} \).

More general, for a full rank lattice \( L \subset \mathbb{R}^d \) with corresponding dual lattice \( L^\perp \) and \( T = (t_1, \ldots, t_d) \in \mathbb{R}^d \) the dual lattice of \( T \cdot L \) is \( \left( \frac{1}{t_1}, \ldots, \frac{1}{t_d} \right) \cdot L^\perp \). However, we can calculate a generator of every full rank lattice as the following statement shows.

**Lemma 2.18.** Let \( L \subset \mathbb{R}^d \) be a full rank lattice with generator \( G \) and \( L^\perp \) the dual lattice corresponding to \( L \). Then \( L^\perp \) is a lattice again and has generator \( (G^{-1})^T \).

**Proof.** Let \( G \) be a generator of \( L \). Since \( L \) has full rank, \( G \) is invertible. We obtain that

\[
\langle m, z \rangle \in \mathbb{Z} \text{ for all } m \in \mathbb{Z}^d \text{ if and only if } z \in \mathbb{Z}^d.
\]

By substituting \( z = G^T y \) we get

\[
L^\perp = \{ y \in \mathbb{R}^d : \langle \ell, y \rangle \in \mathbb{Z} \text{ for all } \ell \in L \}
\]

\[
= \{ y \in \mathbb{R}^d : \langle Gm, y \rangle \in \mathbb{Z} \text{ for all } m \in \mathbb{Z}^d \}
\]

\[
= \{ y \in \mathbb{R}^d : \langle m, G^T y \rangle \in \mathbb{Z} \text{ for all } m \in \mathbb{Z}^d \}
\]

\[
= \{(G^T)^{-1} z : \langle m, z \rangle \in \mathbb{Z} \text{ for all } m \in \mathbb{Z}^d \}
\]

\[
= G^{-T}(\mathbb{Z}^d).
\]

The rank of \( G^{-T} \) is equal to the rank of \( G \). Hence, the columns of \( G^{-T} \) are linearly independent as well and \( L^\perp \) is indeed a lattice. Further \( L^\perp \) has generator \( G^{-T} \).

Lemma 2.18 directly implies some important statements.

**Corollary 2.19.** Let \( L \) be a full rank lattice and let \( L^\perp \) be the corresponding dual lattice. Then

1. \( L^\perp \) is a full rank lattice.
2. \( (L^\perp)^\perp = L \).
3. \( \det L^\perp = \frac{1}{\det L} \).

It would be possible to define the dual lattice \( L^\perp \) as above also for a non-full rank lattice \( L \). But then, in general, \( L^\perp \) would not be a lattice as the following example shows.
Example 2.20. Let $L$ be the lattice generated by

$$G = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and let $L^\perp$ be defined as above. Then

$$L^\perp = \{ z \in \mathbb{R}^2 : \langle z, k \cdot G \rangle \in \mathbb{Z} \text{ for all } k \in \mathbb{Z} \}$$

$$= \{ (z_1, z_2) \in \mathbb{R}^2 : k \cdot (z_1 + z_2) \in \mathbb{Z} \text{ for all } k \in \mathbb{Z} \}$$

$$= \{ (z_1, z_2) \in \mathbb{R}^2 : z_1 + z_2 \in \mathbb{Z} \}.$$

Hence, every point $(z_1, -z_1) \in \mathbb{R}^2$ is an element of $L^\perp$ and $L^\perp$ is not discrete.

Remark 2.21. The additional assumption of full rank in the definition of the dual lattice and some statements may look very strict, but we have to emphasize that for a given lattice $L \subset \mathbb{R}^d$ of rank $n$ there exists a full rank lattice $L' \subset \mathbb{R}^n$ which is isomorphic to $L$.

Lemma 2.22. Let $L \subset \mathbb{R}^d$ be a full rank lattice and let $L^\perp$ be the dual lattice corresponding to $L$. Then the inequality

$$1 \leq \lambda_i(L) \cdot \lambda_{d+1-i}(L^\perp) \leq d!$$

holds for all $i \in \{1, ..., d\}$.

Proof. See [2, Chapter 8; Theorem 6].

For the next theorem we define the set

$$\mathcal{H} = \left\{ T \in \mathbb{R}^d : \prod_{j=1}^d |t_j| = 1 \right\}.$$

For a lattice $L$ with admissibility constant $k$ the lattice $T \cdot L$ with $T \in \mathcal{H}$ has admissibility constant $k$ as well as we can see easily:

$$\inf_{T \cdot L \setminus \{0\}} \prod_{j=1}^d |\ell_j| = \inf_{T \setminus \{0\}} \prod_{j=1}^d |t_j| \cdot \prod_{j=1}^d |\ell_j| = \inf_{T \setminus \{0\}} \prod_{j=1}^d |t_j| \cdot \prod_{j=1}^d |\ell_j| = k.$$

Theorem 2.23. Let $L \subset \mathbb{R}^d$ be an admissible full rank lattice. Then $L^\perp$ is an admissible lattice too.

Proof. We have already shown in Lemma 2.18 that $L^\perp$ is a lattice. For $d = 1$ the statement holds obviously. So let $d \geq 2$.

First we want to prove the following equation:

$$k = d^{-d/2} \cdot \inf_{T \in \mathcal{H}} \lambda_1(T \cdot L)^d, \quad (2.2)$$

where $k$ is the admissibility constant of $L$. We do this by showing that the right hand side is less or equal the left hand side and vice versa.
"≥": There exists an \( m \in L \) such that
\[
k = \inf_{\ell \in L \setminus \{0\}} \prod_{j=1}^{d} |\ell_j| = \prod_{j=1}^{d} |m_j|.
\]

For all \( T = (t_1, \ldots, t_d) \in \mathcal{H} \) also
\[
k = \prod_{j=1}^{d} |t_j \cdot m_j|
\]
holds. If one component of \( m \) is equal to zero, without loss of generality the first component, we choose \( U_t := (t^{d-1}, t^{-1}, \ldots, t^{-1}) \in \mathcal{H} \) for \( t > 0 \) and obtain
\[
\inf_{T \in \mathcal{H}} \lambda_1(T \cdot L) \leq \inf_{t > 0} \lambda_1(U_t \cdot L) \leq \lim_{t \to \infty} \lambda_1(U_t \cdot L) = 0.
\]

If no component of \( m \) is equal to zero, let us define \( U := k^{1/d} \cdot (\frac{1}{m_1}, \ldots, \frac{1}{m_d}) \in \mathcal{H} \). Now we have
\[
U \cdot m = k^{1/d} \cdot (1, \ldots, 1)
\]
and therefore,
\[
\inf_{T \in \mathcal{H}} \lambda_1(T \cdot L)^d \leq \lambda_1(U \cdot L)
\]
\[
= \min_{\ell \in L \setminus \{0\}} |U \cdot \ell|^d
\]
\[
\leq |U \cdot m|^d
\]
\[
= |k|^{d/d} \cdot |(1, \ldots, 1)|^d
\]
\[
= k \cdot d^{d/2}.
\]

"≤": Since \( T \cdot L \) has for \( T \in \mathcal{H} \) also admissibility constant \( k \) and due to the inequality of arithmetic and geometric means,
\[
k = \inf_{\ell \in L \setminus \{0\}} \prod_{j=1}^{d} |\ell_j|
\]
\[
= \inf_{\ell \in T \cdot L \setminus \{0\}} \left( \prod_{j=1}^{d} \ell_j^2 \right)^{1/2}
\]
\[
\leq \inf_{\ell \in T \cdot L \setminus \{0\}} \left( \frac{1}{d} \sum_{j=1}^{d} \ell_j^2 \right)^{d/2}
\]
\[
= d^{-d/2} \cdot \lambda_1(C \cdot L)^d
\]
holds.
The inequalities (2.3) and (2.4) show (2.2).

Now let us assume that \( L^\perp \) is not admissible, i.e. the admissibility constant \( k' \) of \( L^\perp \) is equal to zero. Then by using (2.2) the value \( \lambda_1(T \cdot L^\perp) \) can be chosen arbitrarily small for \( T \in \mathcal{H} \). In other words, for every \( \varepsilon > 0 \) there exists a \( T_\varepsilon \in \mathcal{H} \) such that

\[
\lambda_1(T_\varepsilon \cdot L^\perp) < \varepsilon.
\]

By using Minkowski’s Second theorem it follows that

\[
\lambda_d(T_\varepsilon \cdot L^\perp)^{d-1} \geq \lambda_d(T_\varepsilon \cdot L^\perp) \cdot \prod_{j=2}^{d-1} \lambda_j(T_\varepsilon \cdot L^\perp)
\]
\[
\geq \frac{2^d \cdot \det L^\perp}{d! \cdot \text{vol}_n(B(0,1))} \cdot \frac{1}{\lambda_1(T_\varepsilon \cdot L^\perp)}
\]
\[
= \frac{c(d, L)}{\lambda_1(T_\varepsilon \cdot L^\perp)}
\]
\[
> \frac{c}{\varepsilon},
\]

We know that \( T \cdot L^\perp = ((\frac{1}{t_1}, ..., \frac{1}{t_d}) \cdot L)^\perp =: (T^{-1} \cdot L)^\perp \). Therefore, and by using Lemma 2.22, we obtain

\[
\lambda_1((T_\varepsilon \cdot L^\perp)^\perp) = \lambda_1(T_\varepsilon^{-1} \cdot L) \leq \frac{d!}{\lambda_d(T_\varepsilon \cdot L^\perp)} < \frac{\varepsilon^{1/(d-1)}}{d! \cdot \varepsilon^{1/(d-1)}}
\]

for arbitrary \( \varepsilon > 0 \). Since \( T_\varepsilon^{-1} \in \mathcal{H} \) and by (2.2) we have

\[
k \leq \inf_{\varepsilon > 0} \lambda_1(T_\varepsilon^{-1} \cdot L)^d = 0
\]

in contradiction to the assumption. \( \square \)
3 Fourier transform

In this chapter we introduce the Fourier transform, in particular we concentrate on the connection between Fourier transform and lattices. This will lead to the Poisson summation formula and some lemmas related to it.

3.1 General properties

Definition 3.1. Let \( f \in L^1(\mathbb{R}^d) \) and let \( \langle ., . \rangle \) be the Euclidean inner product. Then we define for \( y \in \mathbb{R}^d \)

\[
\hat{f}(y) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi i \langle x, y \rangle) dx
\]

as the Fourier transform of \( f \).

For functions \( f \in L^1(\mathbb{R}^d) \) we denote for \( x \in \mathbb{R}^d \) by \( \tau_x f \) the translation of \( f \) by \( x \) given by

\[
(\tau_x f)(y) = f(y - x).
\]

For lattices \( L \subset \mathbb{R}^d \) we obtain that

\[
g(x) := \sum_{\ell \in L} f(x + \ell) = \sum_{\ell \in L} (\tau_{-x} f)(\ell)
\]

is a periodic function. Under some constraints on the decline of \( f \) the convergence of this sum can get ensured as we will later see in Corollary 3.11. If \( g \) exists we call it the periodization of \( f \) in \( L \).

Further we denote for \( \varepsilon > 0 \) by

\[
f_\varepsilon(y) := \varepsilon^{-d} \cdot f\left(\frac{y}{\varepsilon}\right)
\]

the scaling of \( f \) by \( \varepsilon \).

Corollary 3.2. Let \( f \in L^1(\mathbb{R}^d) \) and let \( x \in \mathbb{R}^d \). Then the following properties hold:

1. \( \hat{f}(y) \cdot \exp(2\pi i \langle y, x \rangle) = (\tau_{-x} f)(y) \).
2. \( \hat{f}_\varepsilon(y) = \hat{f}(\varepsilon \cdot y) \).

Proof. 1. By using the substitution \( w = z - x \) we obtain

\[
\hat{f}(y) \cdot \exp(2\pi i \langle y, x \rangle) = \int_{\mathbb{R}^d} f(z) \exp(-2\pi i \langle z, y \rangle + 2\pi i \langle y, x \rangle) dz
\]

\[
= \int_{\mathbb{R}^d} f(z) \exp(-2\pi i \langle (z - x), y \rangle) dz
\]

\[
= \int_{\mathbb{R}^d} f(w + x) \exp(-2\pi i \langle w, y \rangle) dw
\]

\[
= \int_{\mathbb{R}^d} (\tau_{-x} f)(w) \exp(-2\pi i \langle w, y \rangle) dw
\]

\[
= (\tau_{-x} f)(y).
\]
2. Let \( \varepsilon > 0 \). By using the substitution \( z = \frac{x}{\varepsilon} \) we get

\[
\hat{f}_\varepsilon(y) = \int_{\mathbb{R}^d} f_\varepsilon(x) \exp(-2\pi i \langle x, y \rangle) dx
\]

\[
= \int_{\mathbb{R}^d} \varepsilon^{-d} f(\frac{x}{\varepsilon}) \exp(-2\pi i \langle x, y \rangle) dx
\]

\[
= \int_{\mathbb{R}^d} f(z) \exp(-2\pi i \langle z \cdot \varepsilon, y \rangle) dx
\]

\[
= \int_{\mathbb{R}^d} f(z) \exp(-2\pi i \langle z, \varepsilon \cdot y \rangle) dx
\]

\[
= \hat{f}(\varepsilon \cdot y).
\]

\[
\square
\]

**Corollary 3.3.** Let \( L \subset \mathbb{R}^d \) be a lattice with generators \( G \) and \( H \), and let \( f : \mathbb{R}^d \to \mathbb{R} \) be a periodic function with \( L \) as period, i.e. for all \( x \in \mathbb{R}^d \) and \( \ell \in L \) the identity \( f(x) = f(x + \ell) \) holds. Then we have

\[
\int_{\mathcal{P}(G)} f(x) dx = \int_{\mathcal{P}(H)} f(x) dx.
\]

**Proof.** By Lemma 2.7 we know that there exists a unimodular matrix \( U \) such that \( G = HU \). Unimodular matrices \( U = (u_1 \mid \ldots \mid u_d) \) are the composition of column permutations, column multiplications with \(-1\) and the addition of integer multiples of one column to another one applied to the identity matrix. Hence, we can consider these cases separately.

1. Let \( U \) be a permutation matrix. Then \( U \) changes only the order of the basic vectors and \( \mathcal{P}(G) = \mathcal{P}(H) \).

2. Let \( U \) be a matrix which changes sign. In this case \( U \) translates the fundamental cell and due to the periodicity of \( f \) this does not effect the integral.

3. \( u_i \to u_i + k \cdot u_j \) for \( k \in \mathbb{Z}, i \neq j \): In this case we have to cut and translate some pieces of \( GU \). However, the volume of \( \mathcal{P}(GU) \) is equal to the volume of \( \mathcal{P}(G) \) and again due to the periodicity of \( f \) this does not effect the integral.

\[
\square
\]

Hence, in this context we will simply write the fundamental cell of a lattice \( L \) instead of referring to a specific choice of a generator.

**Definition 3.4.** Let \( L \subset \mathbb{R}^d \) be a lattice of full rank and let \( \mathcal{P}(L) \) be the fundamental cell of \( L \). Then we define the function space \( C^\infty[L] \) containing all infinitely often differentiable functions on \( \mathbb{R}^d \) which are periodic with \( L \) as period.

Similarly, for \( 1 \leq q < \infty \) we define the function space \( L^q[L] \subseteq L^q(\mathcal{P}(L)) \) containing all periodic functions with \( L \) as period.

On \( L^q[L] \) we define the norm

\[
\|f\|_{L^q[L]} = \frac{1}{(\det L)^{\frac{q}{2}}} \cdot \|f\|_{L^q(\mathcal{P}(L))}.
\]
For lattices $L$ we define an operation similar to the Fourier transform on $\mathbb{R}^d$.

**Definition 3.5.** Let $L \subset \mathbb{R}^d$ be a lattice of full rank and let $1 \leq q < \infty$. For $f \in L^q[L]$ we define the **Fourier transform on $L$** as

$$\tilde{f}(y) = \frac{1}{\det L} \int_{P(L)} f(x) \exp(-2\pi i (x, y)) dx.$$ 

With the Fourier transform on lattices $L$ we can define new function spaces with some useful properties.

**Definition 3.6.** Let $L \subset \mathbb{R}^d$ be a lattice with corresponding dual lattice $L^\perp$ and let $1 \leq q < \infty$. Then we define

$$L^q_0[L] = \{ f \in L^q[L] : \tilde{f}(z) = 0 \text{ if } z \in L^\perp, \prod_{j=1}^d |z_j| = 0 \}$$

and similarly

$$C^\infty_0[L] = \{ \phi \in C^\infty[L] : \tilde{\phi}(z) = 0 \text{ if } z \in L^\perp, \prod_{j=1}^d |z_j| = 0 \}.$$ 

We mention that $C^\infty_0[L]$ is a dense subset of $L^q_0[L]$ for all $1 \leq q < \infty$.

If the lattice $L$ is admissible in the definition above, $L^\perp$ is admissible as well and the point $z = 0$ is the only point in $L^\perp$ which fulfills $\prod_{j=1}^d |z_j| = 0$. Hence, the defined spaces can be written as

$$L^q_0[L] = \{ f \in L^q[L] : \tilde{f}(0) = 0 \}$$

and

$$C^\infty_0[L] = \{ \phi \in C^\infty[L] : \tilde{\phi}(0) = 0 \}.$$ 

**Theorem 3.7 (Carleson’s theorem).** Let $L \subset \mathbb{R}^d$ be a full rank lattice with corresponding dual lattice $L^\perp$, let $1 < q < \infty$ and let $f \in L^q[L]$. Then the identity

$$f(x) = \sum_{z \in L^\perp} \tilde{f}(z) \cdot \exp(2\pi i \langle z, x \rangle)$$

holds for almost every $x \in P(L)$.

**Proof.** We prove the identity (3.1) in $L^2[L]$. It is much more technical to prove (3.1) almost everywhere and for all $1 < q < \infty$. This can be found for example in [5].

Let us define the inner product

$$\langle f, g \rangle_L = \frac{1}{\det L} \int_{P(L)} f(x) \cdot \overline{g(x)} dx$$

on $L^2[L]$. The set of functions $A_L = \{ \exp(2\pi i \langle z, \cdot \rangle) : z \in L^\perp \}$ defines an orthonormal set with respect to $\langle \cdot, \cdot \rangle_L$ as we can see easily: Let $G$ be a generator of $L$. By Lemma 2.18 we know that $G^{-T}$ is a generator of $L^\perp$ and

$$G^T(L^\perp) = \mathbb{Z}^d.$$
Now let $z_1, z_2 \in L^\perp$. We have $z_1 - z_2 \in L^\perp$ and
\[ G^{-T}(z_1 - z_2) = 0 \text{ if and only if } z_1 = z_2. \]
Hence, by using the substitution $x = Gy$ we have
\[
\langle \exp(2\pi i \langle z_1, \cdot \rangle), \exp(2\pi i \langle z_2, \cdot \rangle) \rangle_L = \frac{1}{\det L} \int_{\mathcal{P}(L)} \exp(-2\pi i \langle z_1 - z_2, x \rangle) dx
= \int_{[0,1)^d} \exp(-2\pi i \langle z_1 - z_2, Gy \rangle) dy
= \int_{[0,1)^d} \exp(-2\pi i \langle G^T(z_1 - z_2), y \rangle) dy
= \begin{cases} 0 & \text{if } z_1 \neq z_2, \\ 1 & \text{if } z_1 = z_2. \end{cases}
\]

$A_L$ even forms a complete set of functions: Let $f \in L^2[L]$ and
\[
\langle f, \exp(2\pi i \langle z, \cdot \rangle) \rangle_L = 0 \text{ for all } z \in L^\perp.
\]
Using the substitution $x = Gy$ as before,
\[
\langle f, \exp(2\pi i \langle z, \cdot \rangle) \rangle_L = \int_{[0,1)^d} f(Gy) \exp(-2\pi i \langle G^T z, y \rangle) dy
= \int_{[0,1)^d} f(Gy) \exp(-2\pi i \langle m, y \rangle) dx
= 0
\]
for every $m \in \mathbb{Z}^d$. Due to the Stone-Weierstrass theorem this is the case if and only if $(f \circ G)(x) = 0$ for almost every $x \in [0,1)^d$. The function $f \circ G$ is periodic with $\mathbb{Z}^d$ as period. Hence, $f \circ G$ is vanishing almost everywhere on $\mathbb{R}^d$. $G$ is regular and therefore $f$ is the zero function almost everywhere. This shows that $A_L$ is a complete set of functions in $L^2[L]$ as asserted.

The basic representation of a function $f \in L^2[L]$ predicts coefficients $a_z \in \mathbb{C}$ such that
\[
f = \sum_{g \in A_L} a_z \cdot g = \sum_{z \in L^\perp} a_z \cdot \exp(2\pi i \langle z, \cdot \rangle).
\]
Using this representation we have
\[
\tilde{f}(y) = \frac{1}{\det L} \int_{\mathcal{P}(L)} f(x) \cdot \exp(-2\pi i \langle y, x \rangle) dx
= \langle f, \exp(2\pi i \langle y, \cdot \rangle) \rangle_L
= \sum_{z \in L^\perp} a_z \langle \exp(2\pi i \langle z, \cdot \rangle) \rangle_{L^\perp} \exp(2\pi i \langle y, \cdot \rangle) \rangle_{L^\perp}
= a_y
\]
and (3.1) is shown in $L^2[L]$. \(\square\)
Lemma 3.8. Let \( L \subset \mathbb{R}^d \) be an admissible lattice and let \( M_y(x) = \prod_{j=2}^{d} \phi(2^{-y_j}x_j) \) be such that \( \sum_{\ell \in L} M_{\ell}(x) = 1 \) for a function \( \phi \in C^\infty(\mathbb{R}) \) with \( \phi(0) = 0 \) and \( \lim_{|x| \to \infty} \phi(x) = 0 \), and \( y \in \mathbb{R}^d \). Then there exists a Fourier multiplier \( M_y \) such that
\[
\widehat{(M_y \psi)}(x) = M_y(x) \cdot \hat{\psi}(x).
\]
For this operator
\[
|\phi(x)| \leq \sum_{\ell \in L} |(M_\ell \phi)(x)|
\]
and
\[
\|\phi\|_{L^q(\mathbb{R}^d)} \leq c \cdot \left( \left\| \sum_{\ell \in L} |(M_\ell \phi)(x)|^2 \right\|_{L^1(\mathbb{R}^d)} \right)^{1/2}
\]
holds for \( 1 < q < \infty \) and a constant \( c(d,q) \in \mathbb{R} \).

Proof. A function \( \phi \) with all these properties can be explicitly given, see for example [13, page 218]. So let us assume that \( \phi \) is indeed a function fulfilling them. Then we have by using Carleson’s theorem for the dual lattice \( L^\perp \) of \( L \) and by assumption
\[
\phi(x) = \sum_{z \in L^\perp} \tilde{\phi}(z) \cdot \exp(2\pi i \langle x, z \rangle)
\]
\[
= \sum_{z \in L^\perp} \sum_{\ell \in L} M_{\ell}(z) \tilde{\phi}(z) \cdot \exp(2\pi i \langle x, z \rangle)
\]
\[
= \sum_{z \in L^\perp} \sum_{\ell \in L} \hat{(M_\ell \phi)}(z) \cdot \exp(2\pi i \langle x, z \rangle)
\]
\[
= \sum_{\ell \in L} \left( \sum_{z \in L^\perp} \hat{(M_\ell \phi)}(z) \cdot \exp(2\pi i \langle x, z \rangle) \right)
\]
\[
= \sum_{\ell \in L} \hat{(M_\ell \phi)}(x)
\]
for all \( x \in \mathbb{R}^d \). Therefore,
\[
|\phi(x)| = \left| \sum_{\ell \in L} \hat{(M_\ell \phi)}(x) \right| \leq \sum_{\ell \in L} |(M_\ell \phi)(x)|.
\]
This completes the first part of the proof.

For the second part of the proof we refer to [13, Lemma 5.2].
3.2 Poisson’s summation formula

In this section we prove a crucial statement for changing from sums over full ranked lattices to its corresponding dual lattice. For realizing this we first show Poisson’s summation formula, which handles the lattice $\mathbb{Z}^d$.

**Lemma 3.9.** Let $f \in L^1(\mathbb{R}^d)$. Then the periodization of $f$ in $\mathbb{Z}^d$,

$$\sum_{\ell \in \mathbb{Z}^d} f(x + \ell),$$

converges in $L^1([0,1]^d)$ and

$$\hat{f}(m) = \int_{[0,1]^d} \sum_{\ell \in \mathbb{Z}^d} f(x + \ell) \cdot \exp(2\pi i \langle m, x \rangle) dx$$

for $m \in \mathbb{Z}^d$.

**Proof.** First we show that $\sum_{\ell \in \mathbb{Z}^d} f(x + \ell)$ converges in $L^1([0,1]^d)$. Consider the sets $\{0,1\}^d - \ell_1$ and $\{0,1\}^d - \ell_2$ are disjoint for distinct $\ell_1, \ell_2 \in \mathbb{Z}^d$, and $\bigcup_{\ell \in \mathbb{Z}^d} ([0,1]^d - \ell) = \mathbb{R}^d$. Hence, we obtain

$$\int_{[0,1]^d} \left| \sum_{\ell \in \mathbb{Z}^d} f(x + \ell) \right| dx \leq \sum_{\ell \in \mathbb{Z}^d} \int_{[0,1]^d} |f(x + \ell)| dx$$

$$= \sum_{\ell \in \mathbb{Z}^d} \int_{[0,1]^d} |f(x)| dx$$

$$= \int_{\mathbb{R}^d} |f(x)| dx < \infty.$$

Again by interchanging summation and integration we have

$$\int_{[0,1]^d} \sum_{\ell \in \mathbb{Z}^d} f(x + \ell) \cdot \exp(2\pi i \langle m, x \rangle) dx = \sum_{\ell \in \mathbb{Z}^d} \int_{[0,1]^d - \ell} f(x) \cdot \exp(2\pi i \langle m, x - \ell \rangle) dx$$

$$= \int_{\mathbb{R}^d} f(x) \cdot \exp(2\pi i \langle m, x \rangle) dx$$

$$= \hat{f}(m).$$

\[ \square \]

**Theorem 3.10** (Poisson’s summation formula). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and let $A, \delta > 0$ be some constants such that for all $y \in \mathbb{R}^d$ the bounds

$$|f(y)| \leq A(1 + |y|)^{-d-\delta} \quad \text{and} \quad |\hat{f}(y)| \leq A(1 + |y|)^{-d-\delta}$$

hold. Then

$$\sum_{\ell \in \mathbb{Z}^d} f(x + \ell) = \sum_{\ell \in \mathbb{Z}^d} \hat{f}(\ell) \cdot \exp(2\pi i \langle \ell, x \rangle)$$

(3.2)

holds for all $x \in \mathbb{R}^d$. In particular, both sides converge in $L^1([0,1]^d)$ and uniformly in $x$ and constitute a continuous function in $x$. 

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Proof. First of all we show that $f$ is integrable. By integration in polar coordinates we know that for the normalized surface measure $\sigma$ on the $(d-1)$-dimensional unit sphere $\partial B(0,1)$ of $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} |f(y)|dy \leq A \cdot \int_{\mathbb{R}^d} (1 + |y|)^{-d-\delta}dy$$

$$= A \cdot \int_0^{\infty} \int_{\partial B(0,1)} r^{d-1} \cdot (1 + r)^{-d-\delta}d\sigma dr$$

$$= A\sigma(\partial B(0,1)) \cdot \int_0^{\infty} r^{d-1} \cdot (1 + r)^{-d-\delta}dr$$

$$\leq A\sigma(\partial B(0,1)) \cdot \int_0^{\infty} (1 + r)^{-1-\delta}dr$$

$$\leq A\sigma(\partial B(0,1)) \cdot \sum_{k\in\mathbb{N}} \frac{1}{(1 + k)^{1+\delta}}$$

$$< \infty$$

holds. Now let us show the absolute convergence of the Fourier series in (3.2). Due to our assumption on $\hat{f}$ we have

$$\sum_{\ell \in \mathbb{Z}^d} |\hat{f}(\ell) \cdot \exp(2\pi i \langle \ell, x \rangle)| = \sum_{\ell \in \mathbb{Z}^d} |\hat{f}(\ell)| \leq \sum_{\ell \in \mathbb{Z}^d} A(1 + |\ell|)^{-d-\delta}.$$ 

We reorder the last sum, which is allowed since it has only positive summands, and sum up over the hull of the $(d-1)$-dimensional centred cubes with edge length $2k$ for $k \in \mathbb{N}$.

Figure 6: After reordering the sum in (3.3) we sum up over the hull of the $(d-1)$-dimensional centred cubes with edge length equal to 0, 1, 2, ...
For the upper bound we take the sum over all hypersurfaces, i.e. we use the points on the edges of the cubes several times. Each hypersurface contains \((2k + 1)^{d-1}\) many points. We know that each of these cubes contain \(2d\) many hypersurfaces. Hence, we have for the amount of points in each summand the upper bound \(2d(2k + 1)^{d-1}\). Since \(|z| \geq ||z||_\infty\), we obtain the upper bound

\[
A \cdot \sum_{\ell \in \mathbb{Z}^d} \frac{1}{(1 + ||\ell||)^{d+\delta}} \leq A \cdot \left( \sum_{k \in \mathbb{N}} \frac{2d(2k + 1)^{d-1}}{(1 + k)^{d+\delta}} \right) \\
\leq 2dA \cdot \left( \sum_{k \in \mathbb{N}} \frac{2^{d-1}(k + 1)^{d-1}}{(1 + k)^{d+\delta}} \right) \\
\leq 2^d dA \cdot \sum_{k \in \mathbb{N}} \frac{1}{(1 + k)^{1+\delta}} < \infty.
\]

(3.3)

Therefore, the Fourier series

\[
g_1(x) := \sum_{\ell \in \mathbb{Z}^d} \hat{f}(\ell) \cdot \exp(2\pi i \langle \ell, x \rangle)
\]

converges in \(L^1([0,1]^d)\) and uniformly in \(x\). Hence, \(g_1\) is continuous and has Fourier coefficients

\[
\hat{g}_1(m) = \hat{f}(m)
\]

for \(m \in \mathbb{Z}^d\). Using Lemma 3.9, we obtain that

\[
g_2(x) := \sum_{\ell \in \mathbb{Z}^d} f(x + \ell)
\]

converges in \(L^1([0,1]^d)\). Similarly as above, we obtain that \(g_2\) converges uniformly in \(x\). Hence, \(g_2\) is also continuous and, by Lemma 3.9 again, has Fourier coefficients

\[
\hat{g}_2(m) = \hat{f}(m) = \hat{g}_1(m)
\]

for \(m \in \mathbb{Z}^d\). Therefore

\[
g_2 = g_1.
\]

Corollary 3.11. Let \(L \subset \mathbb{R}^d\) be a full rank lattice with the corresponding dual lattice \(L^\perp\) and let \(f : \mathbb{R}^d \to \mathbb{R}\) be a continuous function fulfilling

\[
|f(y)| \leq A(1 + |y|)^{-d-\delta} \quad \text{and} \quad |\hat{f}(y)| \leq A(1 + |y|)^{-d-\delta}
\]

(3.4)

for some constants \(A, \delta > 0\). Then

\[
\det L \cdot \sum_{\ell \in L} f(x + \ell) = \sum_{z \in L^\perp} \hat{f}(z) \cdot \exp(2\pi i \langle z, x \rangle)
\]

(3.5)

for all \(x \in \mathbb{R}^d\). In particular,

\[
\det L \cdot \sum_{\ell \in L} f(\ell) = \sum_{z \in L^\perp} \hat{f}(z)
\]

holds and the Fourier series in both equations converge.
Proof. Let \( G \) be a generator of the lattice \( L \) such that \( \det G = \det L \) and define
\[
g(y) = f(Gy)
\]
for \( y \in \mathbb{R}^d \). Since \( G \) is regular, we know that \( g \in L^1(\mathbb{R}^d) \) and that there exists a \( c \in \mathbb{R} \) such that
\[
|y| \leq c \cdot |Gy|.
\]
We show that \( g \) fulfills (3.4) for some constants \( \tilde{A}, \tilde{\delta} > 0 \). We choose \( \tilde{A} = A \cdot \min(1, c) \) and \( \tilde{\delta} = \delta \). Hence,
\[
|g(y)| = |f(Gy)| \\
\leq A(1 + |Gy|)^{-d-\delta} \\
\leq A(1 + c \cdot |y|)^{-d-\delta} \\
\leq A \cdot \min(1, c) \cdot (1 + |y|)^{-d-\delta} \\
= \tilde{A}(1 + |y|)^{-d-\delta}.
\]
Similarly, we obtain
\[
|\hat{g}(y)| = |\hat{f}(Gy)| \leq \tilde{A}(1 + |y|)^{-d-\delta}.
\]
Therefore \( g \) fulfills the preconditions for Poisson’s summation formula and by using (3.2) we obtain
\[
\sum_{m \in \mathbb{Z}^d} g(m + y) = \sum_{m \in \mathbb{Z}^d} \hat{g}(m) \cdot \exp(2\pi i (m, y)) \tag{3.6}
\]
for all \( y \in \mathbb{R}^d \). Due to Lemma 2.18, \( G^{-T} \) is a generator of the dual lattice \( L^\perp \). Since \( G \) has full rank we can use the substitution \( x = Gy \) to obtain
\[
\hat{f}(G^{-T}m) = \int_{\mathbb{R}^d} f(x) \exp(-2\pi i (G^{-T}m, x)) dx \\
= \int_{\mathbb{R}^d} f(x) \exp(-2\pi i (m, G^{-1}x)) dx \\
= \det G \cdot \int_{\mathbb{R}^d} f(Gy) \exp(-2\pi i (m, y)) dy \\
= \det L \cdot \hat{g}(m).
\]
for \( m \in \mathbb{Z}^d \). Using this identity, we continue (3.6) for every \( x \in \mathbb{R}^d \) by choosing \( y = G^{-1}x \) with
\[
\det L \cdot \sum_{\ell \in L} f(x + \ell) = \det L \cdot \sum_{m \in \mathbb{Z}^d} f(x + Gm) \\
= \det L \cdot \sum_{m \in \mathbb{Z}^d} g(m + y) \\
= \det L \cdot \sum_{m \in \mathbb{Z}^d} \hat{g}(m) \cdot \exp(2\pi i (m, y)) \\
= \sum_{m \in \mathbb{Z}^d} \hat{f}(G^{-T}m) \cdot \exp(2\pi i (m, y)) \\
= \sum_{z \in L^\perp} \hat{f}(z) \cdot \exp(2\pi i (z, x)).
\]
\( \square \)
Convolution and support of a function

Strongly connected to the Fourier transform are the convolution and the support of functions. In this short paragraph we want to define them and state the simplest properties, which are easy to prove.

**Definition 3.12.** Let \( f, g \in L^1(\mathbb{R}^d) \). Then we define for \( y \in \mathbb{R}^d \)

\[
f \ast g(y) = \int_{\mathbb{R}^d} f(x)g(y - x)dx
\]

as the convolution of \( f \) and \( g \).

**Corollary 3.13.** Let \( f, g \in L^1(\mathbb{R}^d) \). Then

1. \( f \ast g = g \ast f \).

If further \( g \in C^k(\mathbb{R}^d) \) with bounded \( D^\alpha g \) for all \( |\alpha| \leq k \), then

2. \( \frac{\partial}{\partial x_i}(f \ast g) = \frac{\partial f}{\partial x_i} \ast g \).

3. \( f \ast g \in C^k(\mathbb{R}^d) \).

**Definition 3.14.** Let \( f : A \subseteq \mathbb{R}^d \to \mathbb{R} \). Then we denote by

\[
\text{supp}(f) = \{x \in A : f(x) \neq 0\}
\]

the support of \( f \).

**Corollary 3.15.** Let \( f, g \in L^1(\mathbb{R}^d) \) and let \( \varepsilon > 0 \). Then

1. \( \text{supp}(f \ast g) \subseteq \text{supp}(f) + \text{supp}(g) \).

2. \( \text{supp}(f_\varepsilon) = \frac{1}{\varepsilon} \cdot \text{supp}(f) \).
4 Discrepancy

In this chapter we first define measures for a given set of points to assess how good these points are distributed, namely the extreme and the $L^q$-star discrepancy. Afterwards we try to measure admissible lattices and also shifted lattices in this sense. Skriganov’s theorem will lead us to some upper bounds which we can use to obtain nice bounds of the discrepancies of these point sets.

4.1 Extreme and $L^q$-star discrepancy

For axis-parallel boxes in $\mathbb{R}^d$ let us use the notation $[a, b] = [a_1, b_1] \times \ldots \times [a_d, b_d]$.

**Definition 4.1.** Let $P = \{x_0, \ldots, x_{N-1}\}$ be a set of points in $[0, 1]^d$, let $A(\Omega; P)$ be the number of points in $P \cap \Omega$ and let $J$ be the set of axis-parallel boxes in $[0, 1]^d$. Then we denote by

$$D_N(P) = \sup_{\Omega \in J} \left| \frac{A(\Omega; P)}{N} - \text{vol}_d(\Omega) \right| = \sup_{a, b \in [0, 1]^d} \left| \frac{A([a, b]; P)}{N} - \text{vol}_d([a, b]) \right|$$

the extreme discrepancy of $P$.

$A(\Omega; P)$ and $J$ can be explicitly stated as

$$A(\Omega; P) = |\Omega \cap P| = \sum_{i=0}^{N-1} 1_\Omega(x_i),$$

and

$$J = \{ T \cdot [0, 1]^d + x \subseteq [0, 1]^d : T, x \in \mathbb{R}^d \}.$$

**Definition 4.2.** Let $P = \{x_0, \ldots, x_{N-1}\}$ be a set of points in $[0, 1]^d$. Then we denote by

$$D_N^*(P) = \sup_{a \in [0, 1]^d} \left| \frac{A([0, a]; P)}{N} - \text{vol}_d([0, a]) \right|$$

the star discrepancy of $P$. 

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Obviously for a set of points $P = \{x_0, ..., x_{N-1}\}$ in $[0,1]^d$

$$D_N^*(P) \leq D_N(P)$$

holds. However, we can also bound the extreme discrepancy in terms of the star discrepancy as the following lemma shows:

**Lemma 4.3.** Let $P = \{x_0, ..., x_{N-1}\}$ be a set of points in $[0,1]^d$. Then

$$D_N(P) \leq 2^d \cdot D_N^*(P)$$

holds.

**Proof.** In the definition of the extreme and star discrepancy the supremum could also be taken over all half-open rectangles, i.e.

$$D_N(P) = \sup_{a,b \in [0,1]^d} \left| \frac{A([a,b]; P)}{N} - \text{vol}_d([a,b]) \right|$$

and

$$D_N^*(P) = \sup_{a \in [0,1]^d} \left| \frac{A([0,a]; P)}{N} - \text{vol}_d([0,a]) \right|.$$
Hence,
\[ A([a, b]; P) = A([0, b]); P) - A([0, a_1] \times [0, b_2]; P) - A([0, b_1] \times [0, a_2]; P) + A([0, a]; P) \]
and therefore
\[ D_N(P) \leq 4 \cdot D_N^*(P). \]

Hence, it is sufficient to give upper bounds for the star discrepancy to receive upper bounds for the extreme discrepancy and vice versa.

**Definition 4.4.** Let \( 1 \leq q < \infty \) and let
\[ \delta_P(a) := \frac{A([0, a]; P) - \text{vol}_d([0, a])}{N}. \] (4.1)
Then we denote by
\[ \Delta^q_N(P) = \|\delta_P\|_{L^q([0, 1]^d)} = \left( \int_{[0, 1]^d} \left| \frac{A([0, a]; P) - \text{vol}_d([0, a])}{N} \right|^q da \right)^{1/q} \]
the \( L^q \)-star discrepancy of \( P \).

The star discrepancy of a point set \( P \) can be seen as the \( L^\infty \)-star discrepancy of \( P \).

For every point set \( P \) in \([0, 1]^d\) and \( q \in [1, \infty) \) we have
\[ \Delta^q_N(P) = \left( \int_{[0, 1]^d} \left| \frac{A([0, a]; P) - \text{vol}_d([0, a])}{N} \right|^q da \right)^{1/q} \]
\[ \leq \left( \int_{[0, 1]^d} \sup_{t \in [0, 1]^d} \left| \frac{A([0, t]; P) - \text{vol}_d([0, t])}{N} \right|^q da \right)^{1/q} \]
\[ = \sup_{t \in [0, 1]^d} \left| \frac{A([0, t]; P) - \text{vol}_d([0, t])}{N} \right| \]
\[ = D^*_N(P). \]

We denote for a lattice \( L \subset \mathbb{R}^d \) by \( D(L) \) the extreme discrepancy of the point set \( L \cap [0, 1]^d \) and by \( \Delta^q(L) \) the \( L^q \)-star discrepancy of \( L \cap [0, 1]^d \). Further,
\[ A(\Omega; L) = \sum_{\ell \in L} 1_\Omega(\ell) \] for axis-parallel boxes \( \Omega \).
Similarly, we define for \( T, x \in \mathbb{R}^d \) the operations \( D(L_T, x), L^q(L_T, x) \) and \( A(\Omega; L_T, x) \).
If \( T \) has no vanishing component, we write \( T^{-1} := (\frac{1}{t_1}, \ldots, \frac{1}{t_d}) \) and see that
\[ A(\Omega_T, x; L) = A(T \cdot \Omega, L - x) = A(\Omega; T^{-1} \cdot (L - x)) = A(\Omega; L_{T^{-1} \cdot (L - x)}). \]

Further we have the identity
\[ \text{vol}_d([a, b]_{T, x}) = \text{vol}_d([a, b]_{T, 0}) = \prod_{j=1}^d (b_j - a_j)|t_j|. \]
4.2 Discrepancy of admissible lattices

In this section we give some bounds for the $L^q$-star discrepancy and star discrepancy of shifted lattices. Therefore we show Skriganov’s theorem, which is fundamental for this aim.

Lemma 4.5. Let $L \subset \mathbb{R}^d$ be an admissible lattice with admissibility constant $k$. Then there exists a constant $c(k, \det L)$ such that for every $t > 0$ and $x \in \mathbb{R}^d$

$$A(t \cdot [0,1]^d + x; L) \leq c \cdot (1 + t)^{2d^2}$$

holds.

Proof. First let us define the function

$$H(x) = \prod_{i=1}^{d} (1 + |x_i|)^{-2d}.$$  

We obtain the bound

$$H(x) \leq (1 + |x|)^{-2d}$$  \hspace{1cm} (4.2)

and for every lattice $L \subset \mathbb{R}^d$

$$\sup_{x \in B(0, r)} H(\ell - x) \leq (1 + r)^{2d^2} H(\ell),$$

since $H$ is monotonically in each component. Further we have

$$\sum_{\ell \in L} H(\ell) < \infty.$$

To prove this, let first $L = \mathbb{Z}^d$. By using the bound (4.2) we obtain

$$\sum_{m \in \mathbb{Z}^d} H(m) \leq \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-2d}.$$

As we have already seen in the proof of Poisson’s summation formula in detail, the upper bound

$$\sum_{m \in \mathbb{Z}^d} (1 + |m|)^{-2d} \leq 2^{d^2} \cdot \sum_{k \in \mathbb{N}} \frac{1}{(1 + k)^{d+1}} < \infty$$

is valid for $d \geq 1$. Since a generator $G$ of $L$ is regular, we know that there exists a constant $\tilde{c} \in \mathbb{R}$ such that $|z| \leq \tilde{c} \cdot |Gz|$ for all $z \in \mathbb{R}^d$. Therefore,

$$\sum_{\ell \in L} H(\ell) = \sum_{m \in \mathbb{Z}^d} H(Gm) \leq \sum_{m \in \mathbb{Z}^d} (1 + \tilde{c} \cdot |m|)^{-2d} \leq \min(1, \tilde{c}) \cdot \sum_{m \in \mathbb{Z}^d} H(m) < \infty.$$

Now let us define the periodic function

$$H_x(L) = \sum_{\ell \in L} H(\ell - x).$$
By using the inequalities above we know that \( H_x(L) \) really exists. If we choose \( r \) big enough such that \( P(L) \subseteq B(0,r) \), we have

\[
\sup_{x \in \mathbb{R}^d} H_x(L) = \sup_{x \in P(L)} H_x(L) = \sup_{x \in B(0,r)} H_x(L) \leq \sum_{\ell \in L} \sup_{x \in B(0,r)} H(\ell - x) \leq (1 + r)^{2d^2} \sum_{\ell \in L} H(\ell) \leq c < \infty
\]

for a \( c(k, \det L) \in \mathbb{R} \). The choice of \( r \) implies the dependency of \( c \) to \( k \) and \( \det L \).

Since \( \min_{x \in t \cdot [0,1]^d} H(x) = (1 + t)^{-2d^2} \) we also know that

\[
1_{t \cdot [0,1]^d}(x) \leq (1 + t)^{2d^2} \cdot H(x).
\]

Hence, if we sum up all lattice points and define \( \Omega = t \cdot [0,1]^d + x \) we obtain

\[
A(\Omega; L) = \sum_{\ell \in L} 1_{\Omega}(\ell) = \sum_{\ell \in L} 1_{t \cdot [0,1]^d}(\ell - x) \leq \sum_{\ell \in L} (1 + t)^{2d^2} \cdot H(\ell - x) = (1 + t)^{2d^2} \cdot H_x(L).
\]

If we combine the inequalities (4.3) and (4.4) the bound

\[
A(\Omega; L) \leq c \cdot (1 + t)^{2d^2}
\]

follows and the statement is proven. \( \Box \)

Now we want to state further properties of the function \( H \), which we need for the proof of Skriganov’s theorem later on.

For every \( y \in \mathbb{R}^d \) it holds that

\[
|y| \leq \sqrt{d \cdot \max_{j=1,\ldots,d} y_j^2} = \sqrt{d \cdot ||y||_\infty} \leq 2d ||y||_\infty.
\]

For \( H \) we have the upper bound

\[
H(x) = \prod_{j=1}^{d} \frac{1}{(1 + |x_j|)^{2d}} \leq \prod_{j=1}^{d} \min \left\{ 1, \frac{1}{|x_j|^{2d}} \right\} \leq \min \left\{ 1, \frac{1}{||x||_\infty^{2d}} \right\}.
\]

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If we combine these two inequalities, we obtain for $p > 0, z \in \mathbb{R}^d$

$$H((2^{-z_1}, ..., 2^{-z_d})/p) \leq \min \left\{ 1, \frac{p^{2d}}{p^{2d}} \right\}$$

$$\leq \min \left\{ 1, \frac{2^d}{p^{2d}} \right\}$$

$$\leq \min \left\{ 1, \frac{p^{2d}}{2^{2d}} \right\}$$

$$\leq \min \left\{ 1, \frac{1}{p^{2d}} \right\}.$$

We now want to consider the sum

$$\sum_{\ell \in L} H((2^{z_1-\ell_1}, ..., 2^{z_d-\ell_d})/p)$$

for $p > 0$ and a lattice point $z \in L$. The convergence of this sum follows by the exponential decrease of the summands. However, we can give more explicitly an upper bound:

We know that there exists a radius $r \in \mathbb{R}^+$ such that

$$\frac{p^{2d}}{2^{2d}} \geq 1 \text{ for all } \ell \in L \text{ with } |\ell - z| < r.$$ 

Let

$$t = r \log_2(2 + p) > 0.$$ 

We have constants $c \in \mathbb{R}$, which is an upper bound for the number of lattice points in $B(-z, r)$ along each line, $c' \in \mathbb{R}$, which is an upper bound of the geometric series

$$\sum_{|z - \ell| \geq r} \frac{1}{2^{d/2}},$$

and a sufficiently large $C \in \mathbb{R}$ such that

$$\sum_{\ell \in L} H((2^{z_1-\ell_1}, ..., 2^{z_d-\ell_d})/p) \leq \sum_{|z - \ell| < r} 1 + \sum_{|z - \ell| \geq r} \frac{p^{2d}}{2^{d/2} |z - \ell|^2}$$

$$\leq cr^{d-1} + \frac{p^{2d}}{2^{d/2}} \cdot \sum_{|z - \ell| \geq r} \frac{1}{2^{d/2} |z - \ell|^2}$$

$$\leq cr^{d-1}(\log_2(2 + p))^{d-1} + \frac{p^{2d}}{(2 + p)^{d/2}} \cdot c'$$

$$\leq C(\ln(2 + p))^{d-1}.$$ 

All the constants are only depending on $\det L$, the admissibility constant $k$ and the dimension $d$.

The previous lemma leads to the following theorem, which implicates the connection between admissible lattices and discrepancies we are looking for.

**Theorem 4.6** (Skriganov). Let $L \subset \mathbb{R}^d$ be an admissible lattice with admissibility constant $k$ and let $1 < q < \infty$. Then there exist constants $c(k, \det L)$ and $c_q(k, \det L)$
such that for every centred axis-parallel box \( \Omega = [-\frac{t_1}{2}, \frac{t_1}{2}] \times \ldots \times [-\frac{t_d}{2}, \frac{t_d}{2}] \) with \( T = (t_1, \ldots, t_d) \in \mathbb{R}^d \)

\[
\sup_{x \in \mathcal{P}(L)} |A(\Omega + x; L) - \frac{\prod_{j=1}^d |t_j|}{\det L}| \leq c \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{d-1}
\]

and

\[
\left( \frac{1}{\det L} \cdot \int_{\mathcal{P}(L)} |A(\Omega + x; L) - \frac{\prod_{j=1}^d |t_j|^q}{\det L} |^q \, dx \right)^{1/q} \leq c_q \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{(d-1)/2}
\]

hold.

**Proof.** First of all we need some crucial statements. For any compact set \( \Pi \subset \mathbb{R}^d \) we define for \( \varepsilon > 0 \) the \( \varepsilon \)-approximations

\[\Pi^\varepsilon^+ = \{ x \in \mathbb{R}^d : ||x - y||_\infty \leq \varepsilon \text{ for a } y \in \Pi \}\]

and

\[\Pi^\varepsilon^- = \{ x \in \Pi : ||x - y||_\infty \geq \varepsilon \text{ for all } y \in \partial \Pi \}.
\]

Obviously we have for all \( \varepsilon > 0 \)

\[\Pi^\varepsilon^- \subset \Pi \subset \Pi^\varepsilon^+
\]

and \( \Pi^\varepsilon^-, \Pi^\varepsilon^+ \) are compact again.

Let \( \psi : \mathbb{R}^d \to \mathbb{R}_0^+ \in C^\infty \) with \( \text{supp}(\psi) \subseteq \bar{B}(0,1) \) and \( \int_{\mathbb{R}^d} \psi(x) \, dx = 1 \).

By using Corollary 3.2, we know that \( \text{supp}(\psi \ast 1_{\Pi^-}) \subseteq \bar{B}(0,\varepsilon) + \Pi^- \subset \Pi \). The second inclusion follows by the fact that \( \bar{B}(0,\varepsilon) \subseteq \{ x \in \mathbb{R}^d : ||x||_\infty \} \). Therefore,

\[\sum_{\ell \in L} (\psi \ast 1_{\Pi^-})(\ell - x) \leq 1_{\Pi}(x) \leq \int_{\Pi} \psi(x - y) \, dy = (\psi \ast 1_{\Pi})(x) \leq (\psi \ast 1_{\Pi^+})(x).
\]

Let \( L \) be an admissible lattice. If we sum up the inequalities from before over \( \ell \in L \) at points \( \ell - x \), we obtain

\[
\sum_{\ell \in L} (\psi \ast 1_{\Pi^-})(\ell - x) \leq \sum_{\ell \in L} 1_{\Pi}(\ell - x) = A(\Pi + x; L) \leq \sum_{\ell \in L} (\psi \ast 1_{\Pi^+})(\ell - x).
\]

There exists for every \( \alpha > 0 \) a constant \( c_\alpha \) such that

\[
|\hat{\psi}_\varepsilon(x)| \leq c_\alpha (1 + \varepsilon |x|)^{-\alpha}.
\]

This is the case, since \( \psi \), and therefore also \( \psi \), is an element of the Schwartz space and the Fourier transform of a Schwartz function is again in the Schwartz space. For
more details we have to refer to [14, Chapter 1].
Then due to Poisson’s summation formula and Corollary 3.2 we have

$$\frac{1}{\det L} \sum_{z \in L^\perp} \psi_\varepsilon \ast \hat{1}_{\Pi^-}(z) \cdot \exp(-2\pi i \langle z, x \rangle)$$

$$= \frac{1}{\det L} \sum_{z \in L^\perp} \hat{\psi}_\varepsilon(z) \cdot \hat{1}_{\Pi^-}(z) \cdot \exp(-2\pi i \langle z, x \rangle)$$

$$\leq A(\Pi + x; L)$$

$$\leq \frac{1}{\det L} \sum_{z \in L^\perp} \hat{\psi}_\varepsilon(z) \cdot \hat{1}_{\Pi^-}(z) \cdot \exp(-2\pi i \langle z, x \rangle).$$

Since

$$\hat{\psi}_\varepsilon(0) \cdot \hat{1}_{\Pi^\pm}(0) \cdot \exp(-2\pi i \langle 0, x \rangle) = \hat{\psi}(0) \cdot \int_{\mathbb{R}^d} 1_{\Pi^\pm}(x) dx = \text{vol}_d(\Pi^\pm)$$

and

$$\text{vol}_d(\Pi^-) \leq \text{vol}_d(\Pi) \leq \text{vol}_d(\Pi^+),$$

we have

$$R^-_\varepsilon(x) - \frac{\text{vol}_d(\Pi^+)}{\det L} := \frac{1}{\det L} \sum_{z \in L^\perp \setminus \{0\}} \hat{\psi}_\varepsilon(z) \cdot \hat{1}_{\Pi^-}(z) \cdot \exp(-2\pi i \langle z, x \rangle) - \frac{\text{vol}_d(\Pi^+)}{\det L}$$

$$\leq A(\Pi + x, L) - \frac{\text{vol}_d(\Pi)}{\det L}$$

$$\leq \frac{1}{\det L} \sum_{z \in L^\perp \setminus \{0\}} \hat{\psi}_\varepsilon(z) \cdot \hat{1}_{\Pi^+}(z) \cdot \exp(-2\pi i \langle z, x \rangle) - \frac{\text{vol}_d(\Pi^-)}{\det L}$$

$$= R^+_\varepsilon(x) - \frac{\text{vol}_d(\Pi^-)}{\det L}.$$

Hence,

$$\left| A(\Pi + x, L) - \frac{\text{vol}_d(\Pi)}{\det L} \right| \leq \frac{\text{vol}_d(\Pi^+) - \text{vol}_d(\Pi^-)}{\det L} + |R^-_\varepsilon(x)| + |R^+_\varepsilon(x)|. \quad (4.8)$$

Since this inequality holds for every $x \in \mathbb{R}^d$, we obtain

$$\sup_{x \in P(L)} \left| A(\Pi + x, L) - \frac{\text{vol}_d(\Pi)}{\det L} \right| \leq \frac{\text{vol}_d(\Pi^+) - \text{vol}_d(\Pi^-)}{\det L} + \sup_{x \in P(L)} (|R^-_\varepsilon(x)| + |R^+_\varepsilon(x)|) \quad (4.9)$$

and

$$\left( \int_{P(L)} \left| A(\Pi + x, L) - \frac{\text{vol}_d(\Pi)}{\det L} \right|^q dx \right)^{1/q} \leq \frac{\text{vol}_d(\Pi^+) - \text{vol}_d(\Pi^-)}{\det L} + \|R^-_\varepsilon\|_{L^q(P(L))} + \|R^+_\varepsilon\|_{L^q(P(L))}. \quad (4.10)$$
Now we can start with the actual proof. Let $x \in \mathbb{R}^d$, $\Pi = [-\frac{1}{2}, \frac{1}{2}]^d$ and $T \in \mathbb{R}^d$ with at least one component equal to zero. Let one of those components be $j \in \{1, ..., d\}$. Then there exists a hyperplane given by a value $a \in \mathbb{R}$ such that

$$\Pi T, x \subset \{ z \in \mathbb{R}^d : z_j = a \}.$$ 

By using Corollary 2.16 we know that $A(\Pi T, x; L) \in \{0, 1\}$. Since this holds for every $x \in \mathbb{R}^d$, the inequalities (4.6) and (4.7) hold in this case.

Let $T \in \mathbb{R}^d$ with no vanishing component. We can express $T$ with a $t > 0$ and a $C \in H = \{ C \in \mathbb{R}^d : \prod_{j=1}^d |c_j| = 1 \}$ by $T = t \cdot C$. We already know that $\det(C \cdot L) = \det L$, the lattice $C \cdot L$ has again admissibility constant $k$ and $\text{vol}_d(C \cdot T) = \text{vol}_d(T)$. Therefore, it is sufficient to prove the theorem for $T = (t, ..., t)$.

Further for small $t$ an upper bound can be easily computed by using Lemma 4.5. For big $t$ we choose $\varepsilon$ small enough such that the following statements in the proof hold. For instance let $t > 10$ and choose $\varepsilon = t^{-101}$.

Let $\phi \in C^\infty(\mathbb{R})$ be a non-negative function with $\int_{\mathbb{R}} \phi(x)dx = 1$ which is symmetric around 0 and has supp($\phi$) $\subseteq [-\frac{1}{2}, \frac{1}{2}]$. We define

$$\Phi(x) = \prod_{i=1}^d \phi(x_i).$$

Then $\Phi \in C^\infty(\mathbb{R}^d)$ is also non-negative, has supp($\Phi$) $\subseteq \Pi \subset B(0, 1)$ and $\int_{\mathbb{R}^d} \Phi(x)dx = 1$. Further we know that

$$1_{\Pi/\varepsilon \subset z}(x) = \int_{\mathbb{R}^d} 1_{\Pi/\varepsilon \subset z}(y) \exp(-2\pi i\langle x, y \rangle)dy$$

$$= \prod_{j=1}^d \int_{(\pm \varepsilon - t)/2}^{\pm \varepsilon + t/2} \exp(-2\pi ix_j y_j)$$

$$= \prod_{j=1}^d \exp(\pi i(t \mp \varepsilon)x_j) - \exp(\pi i(-t \mp \varepsilon)x_j).$$

We want to use this knowledge about the terms of $R_{\varepsilon/2}^\pm$ to find some upper bounds. Let us define the function

$$W_{L, \varepsilon}(x) = \sum_{z \in L^\perp \setminus \{0\}} \varepsilon^d k' \cdot \Phi_{\varepsilon}(z) \exp(2\pi i\langle z, x \rangle),$$

where $k'$ is the admissibility constant of the dual lattice $L^\perp$. In [13, Lemma 6.2] we can see that there exists a $y \in \mathbb{R}^d$ such that

$$R_{\varepsilon}^\pm(x) = \frac{1}{\det L} \sum_{j=1}^{2^d} \pm W_{L, \varepsilon}(x + y_j) \frac{1}{(2\pi i)^d}$$

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and $W_{L,\varepsilon} \in C^\infty$ fulfills the assumptions of Lemma 3.8. Hence, we have

$$\sup_{x \in \mathcal{P}(L)} |W_{L,\varepsilon}(x)| \leq \sup_{x \in \mathcal{P}(L)} \sum_{\ell \in L} |(M_\ell W_{L,\varepsilon})(x)| \leq \sum_{\ell \in L} \sup_{x \in \mathcal{P}(L)} |(M_\ell W_{L,\varepsilon})(x)|$$

and

$$||W_{L,\varepsilon}||_{\mathcal{P}(L)} \leq c_q \cdot \left( \sum_{\ell \in L} |(M_\ell W_{L,\varepsilon})(x)|^2 \right)^{1/2} \left\| \sum_{\ell \in L} \left( M_\ell W_{L,\varepsilon} \right)(x) \right\|_{L^q(\mathcal{P}(L))}.$$

We have already seen that the operator $M_\ell$ can be rewritten in terms of $M_0$. Due to Poisson’s summation formula, we can find a connection to the function $H$ defined in Lemma 4.5 and continue with

$$\sum_{\ell \in L} \sup_{x \in \mathcal{P}(L)} |(M_\ell W_{L,\varepsilon})(x)| \leq c \cdot \sum_{\ell \in L} H(\varepsilon^{-1} \cdot (2^{-\ell_1}, \ldots, 2^{-\ell_d})),$$

respectively

$$\left\| \sum_{\ell \in L} |(M_\ell W_{L,\varepsilon})(x)|^2 \right\|_{L^q(\mathcal{P}(L))}^{1/2} \leq c_q^{'} \cdot \left( \sum_{\ell \in L} H(\varepsilon^{-1} \cdot (2^{-\ell_1}, \ldots, 2^{-\ell_d})) \right)^{1/2},$$

for constants $c, c_q \in \mathbb{R}$ depending on the admissibility constant and the determinant of $L$. For details we have to refer to [13, Lemma 6.2] again. As we have seen in (4.5) the upper bounds

$$c \cdot \sum_{\ell \in L} H(\varepsilon^{-1} \cdot (2^{-\ell_1}, \ldots, 2^{-\ell_d})) \leq cC(\ln(2 + \varepsilon))^{d-1} \leq cC(\ln(2 + \varepsilon))^{d-1}$$

and

$$c_q \cdot \left( \sum_{\ell \in L} H(\varepsilon^{-1} \cdot (2^{-\ell_1}, \ldots, 2^{-\ell_d})) \right) \leq c_q C \cdot (\ln(2 + \varepsilon))^{(d-1)/2} \leq c_q C \cdot (\ln(2 + \varepsilon))^{(d-1)/2}$$

hold for a $C(\det L, k) \in \mathbb{R}$. To conclude, we obtain the bounds

$$\sup_{x \in \mathcal{P}(L)} R_{L,\varepsilon}^\pm \leq \frac{1}{(2\pi)^d \det L} \cdot \sum_{j=1}^{2^d} \sup_{x \in \mathcal{P}(L)} |W_{L,\varepsilon}(x)|$$

$$= \frac{1}{\pi^d \det L} \cdot \sup_{x \in \mathcal{P}(L)} |W_{L,\varepsilon}(x)|$$

$$\leq \frac{cC(\ln(2 + \varepsilon))^{d-1}}{\pi^d \det L},$$

respectively

$$||R_{L,\varepsilon}^\pm||_{L^q(\mathcal{P}(L))} \leq \frac{1}{(2\pi)^d \det L} \cdot \sum_{j=1}^{2^d} ||W_{L,\varepsilon}||_{L^q(\mathcal{P}(L))}$$

$$\leq \frac{c_q C(\ln(2 + \varepsilon))^{(d-1)/2}}{\pi^d \det L}.$$
Further we have
\[ 0 \leq \text{vol}_d(\Pi^{\varepsilon/2+}) - \text{vol}_d(\Pi^{\varepsilon/2-}) = (t + \varepsilon)^d - (t - \varepsilon)^d \leq c_d \cdot t^{-100d} \]
for a \( c_d \in \mathbb{R} \). Now if we plug in those terms into the upper bound (4.9), we have for \( \Omega = \Pi_{T,0} \)
\[
\sup_{x \in \mathcal{P}(L)} \left| A(\Pi_{T,x}; L) - \frac{\text{vol}_d(\Pi_{T,x})}{\det L} \right| = \sup_{x \in \mathcal{P}(L)} \left| A(\Omega + x; L) - \frac{t^d}{\det L} \right| 
\leq \frac{c_d \cdot t^{-100d}}{\det L} + \frac{2cC(\ln(2 + t^d))^{d-1}}{\pi^d \det L}
\]
and
\[
\left| A(\Omega + x; L) - \frac{t^d}{\det L} \right|_{L^2(\mathcal{P}(L))} \leq \frac{c_d \cdot t^{-100d}}{\det L} + \frac{2c'C(\ln(2 + t^d))^{(d-1)/2}}{\pi^d \det L}.
\]
Since \( t \) is chosen as a large number, the second terms in the last two inequalities are dominating and constants \( c(k, \det L), c_q(k, \det L) \) can be easily found such that (4.6) and (4.7) hold.

The supremum in the upper bound (4.6) could of course also be taken over all \( x \in \mathbb{R}^d \), but it is sufficient to consider just the \( x \in \mathcal{P}(L) \) due to the repetitive property of lattices.

In Skriganov’s theorem we have shown that in particular
\[
A(T \cdot [0, 1]^d + x; L) = \prod_{j=1}^d |t_j| \over \det L + O \left( \ln^{d-1} \left( \prod_{j=1}^d |t_j| \right) \right)
\]
holds for \( \prod_{j=1}^d \to \infty \).

However, the assumption of admissibility in Skriganov’s theorem is crucial as the following example shows:

**Example 4.7.** Let \( L \) be the non-admissible lattice \( \mathbb{Z}^2 \) and \( \Omega = [-\frac{1}{2}, \frac{1}{2}]^2 \). If we choose \( x = 0 \) and \( T = (t, 1/t) \) for \( t \geq 1 \), then we have \( \prod_{j=1}^2 |t_j| = 1, \det \mathbb{Z}^2 = 1 \) and \( \text{vol}_2(\Omega_{T,0}) = 1 \) for all \( t \). Further we have
\[
A(\Omega_{T,0}; \mathbb{Z}^2) = 2 \cdot [t] - 1.
\]
Hence, the expression
\[
\sup_{x \in \mathcal{P}(L)} \left| A(\Pi_{T,x}) - \frac{\text{vol}_2(\Pi_{T,x})}{\det L} \right| \geq 2 \cdot [t] - 2
\]
is unbounded and there does not exist a constant such that (4.6) in Skriganov’s theorem would hold.
Remark 4.8. The bound (4.7) in Skriganov’s theorem is best possible for all \( d \geq 2 \).
In the case of \( q = 2 \) this follows by the bound of Roth’s theorem, see for example [9, Theorem 2.22], which would be invalid otherwise. For the more general case \( 1 < q < \infty \) we refer to [11].
Moreover, W. M. Schmidt showed in [10] that the bound (4.6) is best possible for \( d = 2 \). For general dimensions \( d > 2 \) it is still not known if this bound is sharp or not.

The following theorem states a connection between admissible lattices and the star discrepancy, respectively \( L^q \)-star discrepancy.

Corollary 4.9. Let \( L \subset \mathbb{R}^d \) be an admissible lattice with admissibility constant \( k \).
Then we have a constant \( c(k, \det L) \) such that for every \( T = (t_1, \ldots, t_d) \in \mathbb{R}^d \) with no vanishing component and \( x \in \mathbb{R}^d \)

\[
N \cdot D^*(T^{-1} \cdot (L - x)) \leq c \cdot \left( \ln \left( 2 + \prod_{j=1}^{d} |t_j| \right) \right)^{d-1},
\]  

(4.11)

where \( N = A(T : [0,1]^d + x, L) \), holds. Further there exists for every \( q \in (1, \infty) \) a constant \( c_q(k, \det L) \) such that for every \( T \in \mathbb{R}^d \) with no vanishing component there is a \( y \in \mathbb{R}^d \) such that

\[
N \cdot \Delta^q(T^{-1} \cdot (L - y)) \leq c_q \cdot \left( \ln \left( 2 + \prod_{j=1}^{d} |t_j| \right) \right)^{(d-1)/2}.
\]  

(4.12)
Proof. Let \( T \in \mathbb{R}^d \) with non zero component and let \( x \in \mathbb{R}^d \). Then we have

\[
N \cdot \delta_{L^{-1},-T^{-1},x}(a) = A([0, a]; L^{-1},-T^{-1},x) - N \cdot \text{vol}_d([0, a])
\]

\[
= A(T \cdot [0, a] + x; L) - \frac{\text{vol}_d(T \cdot [0, a])}{\det L}
\]

\[
+ \frac{\text{vol}_d(T \cdot [0, a])}{\det L} - N \cdot \text{vol}_d([0, a])
\]

\[
= A([0, a]_{T,x}; L) - \frac{\text{vol}_d(T \cdot [0, a])}{\det L}
\]

\[
- \text{vol}_d([0, a]) \cdot \left( A([0, 1]_{T,x}; L) - \frac{\text{vol}_d([0, 1]_{T,x})}{\det L} \right)
\]

First let us prove the bound (4.11). Applying Skriganov’s theorem to both summands we obtain

\[
N \cdot D^*(L^{-1},-T^{-1},x) = N \cdot \sup_{a \in [0,1]^d} |\delta_{L^{-1},-T^{-1},x}(a)|
\]

\[
\leq \sup_{a \in [0,1]^d} \left| A([0, 1]_{a,T,x}; L) - \frac{\text{vol}_d([0, 1]_{a,T,x})}{\det L} \right|
\]

\[
+ \text{vol}_d([0, a]) \cdot \left| A([0, 1]_{T,x}; L) - \frac{\text{vol}_d([0, 1]_{T,x})}{\det L} \right|
\]

\[
\leq \sup_{a \in [0,1]^d} c_1 \cdot \left( \ln \left( 2 + \prod_{j=1}^d a_j |t_j| \right) \right)^{d-1}
\]

\[
+ c_2 \cdot \prod_{j=1}^d a_j \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{d-1}
\]

\[
\leq (c_1 + c_2) \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{d-1}.
\]

To prove (4.12) we use the inequality

\[(a + b)^q \leq 2^{q-1} \cdot (a^q + b^q) \text{ for } a, b \geq 0\]

and obtain

\[
N \cdot (\Delta^q(L^{-1},-T^{-1},x))^q = N \cdot \int_{[0,1]^d} |\delta_{L^{-1},-T^{-1},x}(a)|^q da
\]

\[
\leq 2^{q-1} \cdot \left( \int_{[0,1]^d} \left| A([0, 1]_{a,T,x}; L) - \frac{\text{vol}_d([0, 1]_{a,T,x})}{\det L} \right|^q da \right)
\]

\[
+ \left| A([0, 1]_{T,x}; L) - \frac{\text{vol}_d([0, 1]_{T,x})}{\det L} \right|^q.
\]

If we integrate this expression over the fundamental cell, use Fubini’s formula and
afterwards Skriganov’s theorem we have

$$\frac{N}{\det L} \cdot \int_{\mathcal{P}(L)} (\Delta^q(L_{T^{-1},-T^{-1},x}))^q dx \leq 2^{q-1} \cdot \left( \int_{[0,1]^d} \frac{1}{\det L} \cdot \int_{\mathcal{P}(L)} A([0,1]_{a,T,x}; L) - \frac{\text{vol}_{d}([0,1]_{a,T,x})}{\det L} q \right)^q dxdA$$

$$+ \frac{1}{\det L} \cdot \int_{\mathcal{P}(L)} A([0,1]_{T,x}; L) - \frac{\text{vol}_{d}([0,1]_{T,x})}{\det L} q dx$$

$$\leq 2^{q-1} \cdot \left( \int_{[0,1]^d} c_{q,1}^q \cdot \left( \ln \left( 2 + \prod_{j=1}^d a_j |t_j| \right) \right)^{q(d-1)/2} da \right.$$

$$\left. + c_{q,2}^q \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{q(d-1)/2} \right)$$

$$\leq 2^{q-1} \cdot (c_{q,1}^q + c_{q,2}^q) \cdot \left( \ln \left( 2 + \prod_{j=1}^d |t_j| \right) \right)^{q(d-1)/2}.$$
5 Admissibility of Frolov lattices

In this chapter the main goal is to give a sequence of admissible lattices. To realize this it is necessary to first give a short introduction to symmetric polynomials. We will use some properties of symmetric polynomials to show the admissibility of lattices generated by so-called Frolov matrices. Afterwards we will use some particular Frolov matrices to state the intended sequence of lattices.

5.1 Symmetric polynomials

In this section we deal with the theory of symmetric polynomials, which provides us essential statements for our aims.

Definition 5.1. Let \( A \) be a commutative ring and \( d \geq 1 \). For a monomial \( m = x_1^{e_1} \cdot \ldots \cdot x_d^{e_d} \) with \( e_1, \ldots, e_d \in \mathbb{N}_0 \) the degree of \( m \) is defined as

\[
\deg(m) = e_1 + \ldots + e_d.
\]

Now let \( P = a_1 m_1 + \ldots + a_d m_d \) be a polynomial in \( A[x_1, \ldots, x_d] \) with monomials \( m_1, \ldots, m_d \) and coefficients \( a_1, \ldots, a_d \in A \setminus \{0\} \). Then the degree of \( P \) is defined as

\[
\deg(P) = \max_{\{i=1, \ldots, r\}} \deg(m_i).
\]

Definition 5.2. Let \( A \) be a commutative ring, let \( d \geq 1 \) and let \( P \) be a polynomial in \( A[x_1, \ldots, x_d] \). Then we call \( G \) a symmetric polynomial in \( x_1, \ldots, x_d \), if for all permutations \( \sigma : (1, \ldots, d) \to (1, \ldots, d) \) the equation

\[
G(x_1, \ldots, x_d) = G(x_{\sigma(1)}, \ldots, x_{\sigma(d)})
\]

holds.

Further we call

\[
s_k = \sum_{S \subseteq \{1, \ldots, d\}, |S| = k} \prod_{i \in S} x_i
\]

for \( k \in \{0, \ldots, d\} \) an elementary symmetric polynomial in \( x_1, \ldots, x_d \) with degree \( k \).

Elementary symmetric polynomials in \( x_1, \ldots, x_d \) are symmetric in \( x_1, \ldots, x_d \) and the degree of \( s_k \) in \( A[x_1, \ldots, x_d] \) is equal to \( k \). Now let us consider \( s_k \) as a polynomial in \( A[s_1, \ldots, s_d] \). Then we have to keep in mind, that \( \deg(s_k) = 1 \) in \( A[s_1, \ldots, s_d] \).

Theorem 5.3 (Fundamental theorem of symmetric polynomials). Let \( A \) be a commutative ring and let \( d \geq 1 \). Then each symmetric polynomial \( G \in A[x_1, \ldots, x_d] \) with degree \( n \) has a representation

\[
G(x_1, \ldots, x_d) = H(s_1(x_1, \ldots, x_d), \ldots, s_d(x_1, \ldots, x_d))
\]

for some polynomial \( H \) in \( A[s_1, \ldots, s_d] \), whereas \( s_1, \ldots, s_d \) are the elementary symmetric polynomials in \( x_1, \ldots, x_d \).
Proof. We prove the existence of a polynomial \( H \in A[x_1, \ldots, x_d] \) such that

\[
G(x_1, \ldots, x_d) = H(s_1(x_1, \ldots, x_d), \ldots, s_d(x_1, \ldots, x_d))
\]

by double induction with respect to \( n \) and \( d \). We will use the following strategy:

- **Basis of induction:** We show the statement for both \( d = 1, \) \( n \) arbitrary and \( n = 0, \) \( d \) arbitrary.

- **Induction step:** We assume that the statement holds for both \( d - 1; 0, \ldots, n \) and \( n - 1; 0, \ldots, d \) and show the statement for \( d \) and \( n \).

Let \( d = 1 \). Then every polynomial \( G \in A[x_1] \) with degree \( n \) is symmetric in \( x_1 \). \( s_0 = 1 \) and \( s_1 = x_1 \) are the only elementary symmetric polynomials in \( A[x_1] \). \( G \) can be represented as

\[
G(x_1) = \sum_{i=0}^{n} a_i x_1^i = \sum_{i=0}^{n} a_i \cdot s_1^i = H(s_1)
\]

with coefficients \( a_i \in A, i \in \{0, \ldots, d\} \).

Now let \( n = 0 \), i.e. the polynomial \( G \in A[x_1, \ldots, x_d] \) is constant. Then we can represent \( G \) by choosing \( c \in A \) such that

\[
G(x_1, \ldots, x_d) = c = H(s_1, \ldots, s_d).
\]

Therefore, the basis of induction holds.

Let \( \tilde{s}_1, \ldots, \tilde{s}_{d-1} \) be the elementary symmetric polynomials in \( x_1, \ldots, x_{d-1} \). For them we have the relation

\[
s_i(x_1, \ldots, x_{d-1}, 0) = \tilde{s}_i(x_1, \ldots, x_{d-1}) \quad (5.1)
\]

for \( i \in \{1, \ldots, d - 1\} \).

Now let us continue with the induction step and let \( G \in A[x_1, \ldots, x_d] \), \( d > 1 \), be a symmetric polynomial in \( x_1, \ldots, x_d \) and \( G \) have degree \( n \) in \( A[x_1, \ldots, x_d] \). Consider \( G \) as polynomial in \( x_d \), i.e.

\[
G = \sum_{i=0}^{n} G_i \cdot x_d^i \quad (5.2)
\]

with \( G_i \in A[x_1, \ldots, x_{d-1}] \) for \( i \in \{0, \ldots, d\} \). Due to the isomorphism

\[
(A[x_1, \ldots, x_{d-1}])[x_d] \simeq A[x_1, \ldots, x_d],
\]

this is possible.

\( G_0 \) is symmetric in \( x_1, \ldots, x_{d-1} \) and \( \deg(G_0) \leq \deg(G) = n \) holds. Therefore we can use the first induction hypothesis. Thus, there exists an \( H_0 \in A[x_1, \ldots, x_{d-1}] \) with \( \deg(H_0) \leq n \) such that

\[
G_0(x_1, \ldots, x_{d-1}) = H_0(\tilde{s}_1, \ldots, \tilde{s}_{d-1}).
\]

\[
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\]
We define $H_1$ as symmetric polynomial in $x_1, \ldots, x_d$ such that

$$H_1(x_1, \ldots, x_{d-1}, 0) = H_0(x_1, \ldots, x_{d-1})$$

(5.4)

holds. This is possible due to (5.1).

We define

$$L(x_1, \ldots, x_d) = G(x_1, \ldots, x_d) - H_1(s_1(x_1, \ldots, x_d), \ldots, s_{d-1}(x_1, \ldots, x_d)).$$

For the degree of $L$ we have the estimation\(^3\)

$$\deg(L) \leq \max\{\deg(G), \deg(H_0)\} = \max\{\deg(G), \deg(H_1)\} = n.$$

Hence, by using (5.2) we have

$$L(x_1, \ldots, x_{d-1}, 0) = G_0(x_1, \ldots, x_{d-1}) - H_1(s_1(x_1, \ldots, x_{d-1}, 0), \ldots, s_{d-1}(x_1, \ldots, x_{d-1}, 0))$$

$$= 0.$$

By using Vieta’s formula applied to $L$ we get

$$L(x_1, \ldots, x_d) = r'(x_1, \ldots, x_d) \cdot (x_d - 0)$$

for an $r' \in A[x_1, \ldots, x_d]$. The polynomials $G$ and $H_1$ are symmetric in $x_1, \ldots, x_d$ due to the assumption. Thus, $L$ is also symmetric in $x_1, \ldots, x_d$. Using this symmetry we know that there exists an $r \in A[x_1, \ldots, x_d]$ with $\deg(r) \leq \deg(L) - 1 \leq n - 1$ such that

$$L(x_1, \ldots, x_d) = r(x_1, \ldots, x_d) \cdot x_1 \cdot \ldots \cdot x_d = r(x_1, \ldots, x_d) \cdot s_d(x_1, \ldots, x_d).$$

The second induction hypotheses applied to $r$ provides the existence of an $H_2 \in A[x_1, \ldots, x_d]$ with $\deg(H_1) \leq n - 1$ such that

$$G(x_1, \ldots, x_d) = L(x_1, \ldots, x_d) + H_1(s_1, \ldots, s_{d-1}) = H_2(s_1, \ldots, s_d) \cdot s_d + H_1(s_1, \ldots, s_d)$$

holds. We define the wanted $H$ as $H_2 \cdot s_d + H_1$ and therefore the induction step is shown.

\[\square\]

**Definition 5.4.** We call numbers $\xi_1, \ldots, \xi_d \in \mathbb{R}$ **conjugate elements** over $\mathbb{Q}$, if they are roots of the same minimal polynomial. That means:

There exists a polynomial

$$p(x) = x^d + a_{d-1} \cdot x^{d-1} + \ldots + a_1 \cdot x + a_0$$

with $a_1, \ldots, a_{d-1} \in \mathbb{Z}$ such that

1. $p(\xi_i) = 0$ holds for all $i \in \{1, \ldots, d\}$.

\(^3\)For polynomials $P, Q \in A[x_1, \ldots, x_d]$ it holds that $\deg(P + Q) \leq \max\{\deg(P), \deg(Q)\}$
2. \( p \) is irreducible over \( \mathbb{Q} \), i.e. there exists no polynomial \( q \in \mathbb{Q}[x] \) with \( \deg(q) < d \) and an \( i \in \{1, \ldots, d\} \) such that \( q(\xi_i) = 0 \) holds.

**Lemma 5.5.** Let \( \xi_1, \ldots, \xi_d \in \mathbb{R} \) be conjugate elements over \( \mathbb{Q} \), let \( P \) be a polynomial in \( \mathbb{Z}[x] \) with \( \xi_1, \ldots, \xi_d \) as roots and let \( G(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d] \) be symmetric in \( x_1, \ldots, x_d \). Then

\[
G(\xi_1, \ldots, \xi_d) \in \mathbb{Z}.
\]

**Proof.** \( P \) can be written as

\[
P(x) = \sum_{i=0}^{d} a_i x^i \tag{5.5}
\]

with coefficients \( a_i \) in \( \mathbb{Z} \). By applying Vieta’s formula, \( P \) can also be written as

\[
P(x) = \prod_{j=1}^{d} (x - \xi_j). \tag{5.6}
\]

By expanding the representation (5.5) of \( P \) we obtain

\[
P(x) = \prod_{j=1}^{d} (x - \xi_j)
\]

\[
= x^d + (-1) \cdot \left( \sum_{i=1}^{d} \xi_i \right) \cdot x^{d-1} + \ldots + (-1)^d \cdot \xi_1 \cdot \ldots \cdot \xi_d
\]

\[
= \sum_{i=0}^{d} (-1)^i \cdot x^i \cdot s_{d-i}(\xi_1, \ldots, \xi_d)
\]

\[
= \sum_{i=0}^{d} a_i x^i.
\]

Thus,

\[
a_i = (-1)^i \cdot s_{d-i}(\xi_1, \ldots, \xi_d)
\]

by equating coefficients with (5.6) and therefore \( s_i(\xi_1, \ldots, \xi_d) \in \mathbb{Z} \) for all \( i \in \{1, \ldots, d\} \). As a consequence of the fundamental theorem of symmetric polynomials there exists an \( H \in \mathbb{Z}[x_1, \ldots, x_d] \) such that

\[
G(x_1, \ldots, x_d) = H(s_1(x_1, \ldots, x_d), \ldots, s_d(x_1, \ldots, x_d))
\]

holds. Altogether we get

\[
G(\xi_1, \ldots, \xi_d) = H(s_1(\xi_1, \ldots, \xi_d), \ldots, s_d(\xi_1, \ldots, \xi_d)) \in \mathbb{Z},
\]

because \( H \) has only integer coefficients.

We want to study generators \( B \) of admissible lattices which are Vandermonde matrices with conjugate numbers \( \xi_1, \ldots, \xi_d \) over \( \mathbb{Q} \) as entries:

\[
B = \begin{pmatrix}
1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \xi_d & \xi_d^2 & \cdots & \xi_d^{d-1}
\end{pmatrix}
\]
We call matrices $B$ of this form Frolov matrices and lattices generated by a Frolov matrix Frolov lattice. The admissibility of Frolov lattices is shown by the following theorem.

**Theorem 5.6.** Let $B$ be a Frolov matrix. Then the lattice generated by $B$ is an admissible lattice with admissibility constant equal to 1.

**Proof.** Due to property (2.1) we have to show that the inequality

$$\prod_{j=1}^{d} |(Bm)_j| \geq 1$$

for all $m \in \mathbb{Z}^d \setminus \{0\}$.

Let $m = (m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\}$ be such that $Bm = z$, i.e.

$$z_i = \sum_{j=1}^{d} m_j \xi_i^{j-1}.$$

Hence, the polynomial $\prod_{i=1}^{d} z_i = \prod_{i=1}^{d} (\sum_{j=1}^{d} m_j \xi_i^{j-1})$ is symmetric in $\xi_i$ and has only integer coefficients. Applying Lemma 5.5 we have $\prod_{i=1}^{d} z_i \in \mathbb{Z}$.

It remains to prove $z_i \neq 0$ for $i \in \{1, ..., d\}$. We assume that $z_l = 0$ for an $l \in \{1, ..., d\}$. Then we obtain $\xi_l = 0$.

Let $R_1(x) = \sum_{j=1}^{d} m_j x^{j-1} \in \mathbb{Q}[x]$ be a non-zero polynomial with a root at $\xi_l$. $\mathbb{Q}[x]$ is an Euclidean domain and therefore there exists $R_2$ and $G$ in $\mathbb{Q}[x]$ such that

$$P_d(x) = G(x) \cdot R_1(x) + R_2(x)$$

with $\deg(R_2) < \deg(R_1)$. $R_2$ has a root at $\xi_l$ again:

$$R_2(\xi_l) = P_d(\xi_l) - G(\xi_l)R_1(\xi_l) = 0.$$

If $R_2 \equiv 0$, we get a contradiction to the irreducibility of $P_d$. Otherwise, divide $P_d$ by $P_2$. This way we get a polynomial $P_3$ again with less degree or a contradiction to the irreducibility.

Proceeding with this procedure we get a contradiction in all cases, because the degree function is noetherian and the constant polynomial with a root at $\xi_l$ is constant zero. \qed

We also know the determinant of a Frolov matrix $B$ with entries $\xi_1, ..., \xi_d$. It is given by the formula

$$\det B = \prod_{1 \leq i < j \leq d} (\xi_i - \xi_j).$$

Therefore $B$ is regular if and only if $\xi_1, ..., \xi_d$ are pairwise distinct. Further the entries $\xi_1, ..., \xi_d$ can be placed such that $\det B > 0$. 

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5.2 Example of a Frolov matrix

In this section we give an example of a Frolov matrix for every dimension \( d \). In practice the problem with the method we give is, that it is necessary to compute or actually approximate the roots of a polynomial with degree \( d \). Thus, we present an alternative method by using Chebyshev polynomials. This is possible if the dimension is a power of two.

**Lemma 5.7.** Let \( b_1, \ldots, b_d \in \mathbb{Z} \) be pairwise distinct. Then the polynomial

\[
P(x) = \prod_{j=1}^{d} (x - b_j) - 1
\]

is irreducible over \( \mathbb{Q} \).

**Proof.** We assume that there exist \( p, q \in \mathbb{Z}[x] \) non-constant with \( P(x) = p(x) \cdot q(x) \) and \( \deg(p) = n \leq d, \deg(q) = m \leq d \).

Due to the premise we have

\[P(b_j) = -1 = p(b_j)q(b_j)\]

for \( j \in \{1, \ldots, d\} \) and therefore

\[p(b_j), q(b_j) \in \{-1, +1\} \quad \text{and} \quad p(b_j) = +1 \text{ if and only if } q(b_j) = -1.

Furthermore the leading coefficient of \( P \), denoted by \( \text{lc}(P) \), is equal to 1 and thus

\[\text{lc}(p), \text{lc}(q) \in \{-1, +1\} \quad \text{and} \quad \text{lc}(p) = +1 \text{ if and only if } \text{lc}(q) = +1
\]

holds. According to the fundamental theorem of algebra we have

\[|\{x : p(x) = -1\}| \leq n \quad \text{and} \quad |\{x : p(x) = +1\}| \leq n.
\]

Otherwise \( p \) would be constant in contradiction to our assumption. Altogether \( n \geq d/2 \) holds. Analogously for \( q \) we have \( m \geq d/2 \).

**d odd:** For odd \( d \) we get in these two inequalities even ">". But with the property

\[\deg(p \cdot q) = \deg(P) = d = \deg(p) + \deg(q) = m + n
\]

there is a contradiction to \( m + n > d/2 + d/2 = d \).

That means either \( p \) or \( q \) is constant and therefore invertible (in \( \mathbb{Q} \)). Both, \( p \) and \( q \), were arbitrary and thus \( P \) is by definition invertible.

**d even:** We still have to consider the case \( p \) and \( q \) have degree \( d/2 \) and function value \( -1 \) at \( d/2 \) many points and function value \( +1 \) at \( d/2 \) many points. Otherwise we get a contradiction as before.

W.l.o.g. we have \( \text{lc}(p) = +1 \). The polynomials \( p + 1 \) and \( q - 1 \) can be written as

\[p(x) + 1 = \prod_{j=1}^{d/2} (x - b_j) = q(x) - 1,
\]
because of \( p(b_j) + 1 = 0 = q(b_j) - 1 \) for all \( j \in \{1, \ldots, d\} \). Therefore we get for all \( x \in \mathbb{R} \):

\[
p(x) + 1 = q(x) - 1 \iff p(x) + 2 = q(x)
\]

and thus,

\[
q(x) > p(x).
\]

That is a contradiction to the assumption that \( p(b_j) = 1 \) and \( q(b_j) = -1 \) for \( d/2 \) many points \( j \in \{1, \ldots, d\} \).

Hence, there can not exist \( p, q \in \mathbb{Z}[x] \) non-constant such that \( P(x) = p(x) \cdot q(x) \). Due to Gauss’s lemma there can not exist \( p, q \in \mathbb{Q}[x] \) non-constant with \( P(x) = p(x) \cdot q(x) \) as well and therefore \( P \) is irreducible over \( \mathbb{Q} \).

We want to consider

\[
P_d(x) = \prod_{j=1}^{d} (x - 2j + 1) - 1,
\]

a specific polynomial of the shape used in Lemma 5.7. Applying the lemma, \( P_d \) is irreducible over \( \mathbb{Q} \). Now we want to show that \( P_d \) has real roots with multiplicity 1:

**Proof.** For \( d = 1 \) the statement holds, because \( P_1(x) = x - 2 \) has one root and this is \( x = 2 \). Now let \( d > 1 \).

In this case we use the intermediate value theorem. We are looking for \( d + 1 \) many points in which \( P_d \) is alternately positive and negative. Then there exist \( d \) many distinct points \( \xi_1, \ldots, \xi_d \) with \( P_d(\xi_j) = 0 \).

By easy calculation we obtain

\[
P_d(2l - 1) = -1
\]

for \( l \in \{1, \ldots, d\} \). For the ongoing argumentation we need the following inequality for \( k \in \mathbb{N} \) and \( m \in \mathbb{N} \) odd:

\[
\left| \prod_{j=1}^{d} (4k - 2j + m) \right| \geq 2.
\]

(5.8)

We prove this by using induction with respect to \( d \). For \( d = 2 \) relation (5.8) holds because both factors are non-zero integers and one of them must be greater than one in absolute value.

Now let us assume that (5.8) holds for a fixed \( d \geq 2 \). Then

\[
\left| \prod_{j=1}^{d+1} (2(2k-j)+m) \right| = \left| \prod_{j=1}^{d} (2(2k-j)+1) \cdot |2(2k-d-1)+1| \cdot 2 \cdot |2(2k-d-1)+1| \geq 2
\]

and therefore (5.8) holds also for \( d + 1 \). Hence, (5.8) holds for every \( d \geq 2 \).

We continue with a case distinction for odd and even \( d \):

\(d \) even: Then we have

\[
\lim_{x \to \pm \infty} P_d(x) = +\infty.
\]
Further we have
\[ P_d(4k) = \prod_{j=1}^{d} (4k - 2j + 1) - 1 \]
for \(0 < k < d/2\) and \(k \in \mathbb{N}\). Due to (5.8) we know that \(|P_d(4k)| \geq 1\). The product contains an even number of positive and negative expressions and therefore we get \(P_d(4k) \geq 1 > 0\) for \(d - d/2 - 1 = d/2 - 1\) many points.

With the intermediate value theorem we know about the existence of \(2 + 2(d/2 - 1) = d\) many distinct zero-points. The first addend 2 is a mean value of \(\lim_{x \to \pm\infty} P_d(x)\) and the smallest respectively biggest point where \(P_d\) is equal to \(-1\).

\(d\) odd: Similarly as above we have
\[
\lim_{x \to +\infty} P_d(x) = +\infty, \quad \lim_{x \to -\infty} P_d(x) = -\infty
\]
and
\[ |P_d(4k + 2)| = \left| \prod_{j=1}^{d} (4k - 2j + 3) - 1 \right| \geq 1 > 0 \]
for \(0 \leq k < d/2\) and \(k \in \mathbb{N}\). There is always an odd number of negative factors in \(P_d\) and therefore we have \(d - (d + 1)/2 = (d - 1)/2\) many points where \(P_d\) is positive.

With the additional intermediate value between \(\lim_{x \to +\infty} P_d(x)\) and \(P(1) = 1\) we have \(1 + 2(d-1)/2 = d\) many distinct zero-points.

Graphs of the polynomials \(P_d\) for \(d = 4, 6, 10\) can be found in the appendix.

We have seen that the roots of \(P_d\) are conjugate elements over \(\mathbb{Q}\). With them we can give an example of a Frolov matrix \(F\) which due to Theorem 5.6 generates an admissible lattice with admissibility constant 1.

Further the entries \(\xi_1, ..., \xi_d\) are pairwise distinct as we have seen above. Choosing the entries in a decreasing order the determinant of \(F\) is positive.
Chebyshev lattices

We have seen by using the polynomial \( P_d(x) = \prod_{j=1}^{d} (x - 2j + 1) - 1 \) and their real roots \( \xi_1, \ldots, \xi_d \) as entries one possibility to choose a Frolov matrix in every dimension. One problem with this strategy that occurs is the growth of \( \xi_d \) by choosing bigger \( d \). For dimensions \( d \) equal to a power of two we can use the Chebyshev polynomials

\[
Q_d(x) = 2 \cos \left( d \cdot \arccos \frac{x}{2} \right)
\]

and their real roots

\[
\tilde{\xi}_i = 2 \cos \left( \frac{\pi (2i - 1)}{2d} \right)
\]

for \( i = 1, \ldots, d \) and \( x \in [-2, 2] \). We call the \( d \)-dimensional Frolov matrix with entries \( \tilde{\xi}_1, \ldots, \tilde{\xi}_d \) Chebyshev matrix \( G \) and the lattice generated by \( G \) the Frolov-Chebyshev lattice.

The values \( \tilde{\xi}_i = 2 \cos\left( \frac{\pi (2i - 1)}{2d} \right) \) are for \( i \in \{1, \ldots, d\} \) pairwise distinct and elements in \((-2, 2)\). According to [15, Chapter 4] the polynomials \( Q_d \) are irreducible if and only if \( d = 2^m \) for an \( m \in \mathbb{N} \). For irreducible \( Q_d \) the values \( \tilde{\xi}_1, \ldots, \tilde{\xi}_d \) are conjugated over \( \mathbb{Q} \) and therefore indeed form a Frolov matrix.

In contrast to the roots of the polynomial \( P_d \) the roots of \( Q_d \) are stated explicitly and not just predicted by the Mean value theorem. Hence, the computational error can get decreased by using the roots of \( Q_d \).

In the appendix are graphs of \( Q_d \) for \( d = 4, 6, 10 \) depicted in comparison to the polynomials \( P_d \) of same dimension.
5.3 Sequence of Frolov lattices

In this section we give a sequence of admissible lattices $\mathcal{L}_n$ generated by Frolov matrices $B_n$ given as follows:

Let $B$ be a regular matrix generating an admissible full rank lattice with admissibility constant $k$ such that the determinant of $B$ is positive. For instance, let $B$ be a Frolov matrix. Then we define

$$B_n = \frac{n^{1/d}}{\det B^{1/d}} \cdot B$$

and

$$G_n = (B_n)^{-T} = n^{-1/d} \cdot \left( \frac{1}{\det B^{1/d}} \cdot B \right)^{-T}$$

for $n \in \mathbb{N}$. For $m \in \mathbb{Z}^d$ we have $\prod_{j=1}^d |(B_n m)_j| = \frac{n}{\det B} \cdot \prod_{j=1}^d |(B m)_j|$. By using property (2.1) we obtain

$$\prod_{j=1}^d |(B_n m)_j| \geq \frac{n}{\det B} \cdot k.$$ 

Hence, the lattices $\mathcal{L}_n$ generated by the matrices $B_n$ have admissibility constant $\frac{n \cdot k}{\det B}$. Further, the lattices $\mathcal{L}_n^\perp$ generated by $G_n$ have admissibility constant $\frac{\det B}{n \cdot k}$.

The connection between $\mathcal{L}_1$ and $\mathcal{L}_n$ can be given by

$$\mathcal{L}_n = \frac{n^{1/d}}{\det B^{1/d}} \cdot \mathcal{L}_1.$$ 

We want to use our results for its dual lattice and obtain the relation

$$\mathcal{L}_n^\perp = \frac{\det B^{1/d}}{n^{1/d}} \cdot \mathcal{L}_1^\perp.$$ 

Moreover, by using elementary rules of the determinant we get

$$\det \mathcal{L}_n = \det B_n = n \quad \text{and} \quad \det \mathcal{L}_n^\perp = \det G_n = \frac{1}{n}.$$ 

In Corollary 2.19 we have seen that $\mathcal{L}_n^\perp$ are admissible and therefore provide a sequence of admissible lattices as well.
6 Upper error bound

Let $P = \{x_0, ..., x_{N-1}\}$ be a set of points in $[0,1]^d$ and let $f : [0,1]^d \rightarrow \mathbb{R}$. We consider algorithms

$$Q(f) = \sum_{i=0}^{N-1} a_i \cdot f(x_i)$$

with weights $a_0, ..., a_{N-1} \in \mathbb{R}$. For the choice that all weights are equal to $\frac{1}{N}$ the quadrature formulas are called QMC algorithms and denoted by $Q_P$. Hence we have

$$Q_P(f) = \frac{1}{N} \cdot \sum_{i=0}^{N-1} f(x_i).$$

**Definition 6.1.** Let $f : [0,1]^d \rightarrow \mathbb{R}$ be an integrable function. Then we denote by

$$e(Q,f) = \left| \int_{[0,1]^d} f(t) dt - Q(f) \right|$$

the error of $Q$ applied to $f$.

Let further $(H, \|\cdot\|)$ be a normed vector space with $H$ containing integrable functions $f : [0,1]^d \rightarrow \mathbb{R}$. Then we denote by

$$e(Q,H) = \sup_{f \in H, \|f\| \leq 1} e(Q,f)$$

the worst-case error of $Q$.

As function class $H$ we will start with the set of continuously differentiable functions and generalize to the set of functions of bounded variation and Sobolev spaces in the remaining part of the chapter.

6.1 Continuously differentiable functions

We can compute the error of QMC algorithms applied to continuously differentiable functions in one dimension exactly.

Recall the definition of $\delta_P$ in (4.1).

**Lemma 6.2.** Let $f : [0,1] \rightarrow \mathbb{R}$ be a continuously differentiable function and let $P = \{x_0, ..., x_{N-1}\}$ be a point set in $[0,1]$. Then we have

$$e(Q_P, f) = \left| \int_0^1 f'(t) \cdot \delta_P(t) dt \right|.$$

**Proof.** Since $f$ is continuously differentiable we can apply the fundamental theorem of calculus and have

$$f(x) = f(1) - \int_x^1 f'(t) dt.$$
Hence, we obtain

\[
e(Q_P, f) = \left| \int_0^1 f(x) \, dx - \frac{1}{N} \sum_{i=0}^{N-1} f(x_i) \right|
\]

\[
= \left| \int_0^1 f(x) \, dx - f(1) + \frac{1}{N} \sum_{i=0}^{N-1} 1_{[x_i,1]}(t) \cdot f'(t) \, dt \right|
\]

\[
= \left| - \int_0^1 f'(t) \cdot t \, dt + \frac{1}{N} \sum_{i=0}^{N-1} 1_{[0,t]}(x_i) - t \right| \cdot f'(t) \, dt
\]

\[
= \left| \int_0^1 f'(t) \left( \frac{1}{N} \sum_{i=0}^{N-1} 1_{[0,t]}(x_i) - t \right) \, dt \right|
\]

\[
= \left| \int_0^1 f'(t) \cdot \delta_P(t) \, dt \right|.
\]

By the triangle inequality for integrals and the last lemma we can derive for continuously differentiable \( f : [0, 1] \rightarrow \mathbb{R} \) the upper bound

\[
e(Q_P, f) \leq \int_0^1 |f'(t)| \cdot |\delta_P(t)| \, dt \leq \sup_{x \in [0,1]} |\delta_P(x)| \cdot \int_0^1 |f'(t)| \, dt = ||f'||_{L^1([0,1])} \cdot D^*_N(P).
\]

We also want to state the error of algorithms applied to functions in more dimensions explicitly. This leads to the so-called Hlawka-Zaremba identity.

**Theorem 6.3 (Hlawka-Zaremba identity).** Let \( f \) be a function such that all mixed partial derivatives up to the ordering \( 1 \) in each component exist and are continuous and let \( P = \{x_0, \ldots, x_{N-1}\} \) be a point set in \( [0, 1]^d \). Then we have

\[
e(Q_P, f) = \left| \sum_{\emptyset \neq u \subseteq \{1, \ldots, d\}} (-1)^{|u|} \int_{[0,1]^d} \frac{\partial^{|u|} f}{\partial x_u} (x_u, 1) \cdot \delta_P(x_u, 1) \, dx_u \right|
\]

where \( (x_u, 1) \) is for \( u \subseteq \{1, \ldots, d\} \) the point obtained from \( x \) by replacing the coordinates not in \( u \) by 1.

**Proof.** See [9, Theorem 3.7].

For functions which fulfill the assumption of Theorem 6.3 we have the following error of QMC algorithms.

**Corollary 6.4 (classical Koksma-Hlawka inequality).** Let \( f : [0, 1]^d \rightarrow \mathbb{R} \) be a function with continuous mixed partial derivatives up to order 1, let

\[
||f||_{d, 1} = \sum_{\emptyset \neq u \subseteq \{1, \ldots, d\}} (-1)^{|u|} \int_{[0,1]^d} \left| \frac{\partial^{|u|} f}{\partial x_u} (x_u, 1) \right| \, dx_u
\]
and let $P = \{x_0, ..., x_{N-1}\}$ be a point set in $[0,1]^d$. Then we have

$$e(Q_P, f) \leq ||f||_{d,1} D_N^d(P).$$

**Proof.** By using the Hlawka-Zarneba identity we have

$$e(Q_P, f) \leq \sum_{\emptyset \neq u \subseteq \{1, ..., d\}} |(-1)^{|u|}| \int_{[0,1]^d} \left| \frac{\partial^{|u|} f}{\partial x_u} (x_u, 1) \right| \cdot |\delta_P(x_u, 1)| dx_u$$

$$\leq \sum_{\emptyset \neq u \subseteq \{1, ..., d\}|[0,1]^d} \int \left| \frac{\partial^{|u|} f}{\partial x_u} (x_u, 1) \right| dx_u \cdot \sup_{x \in [0,1]^d} |\delta_P(x)|$$

$$= ||f||_{d,1} \cdot D_N^d(P).$$

$|| \cdot ||_{d,1}$ defines a norm on the set of functions with continuous partial derivatives up to order 1.

In the remaining part of this chapter we want to find a broader class of functions with a similar integration error bound.
6.2 Functions of bounded variation in one dimension

We want to give an upper bound for the integration error of QMC algorithms applied to functions of bounded variation. For one dimension we have the following definition:

**Definition 6.5.** Let \([a, b] \subset \mathbb{R}\) be an interval, let \(f : [a, b] \mapsto \mathbb{R}\) and let \(\mathcal{P}([a, b])\) be the set of partitions of \([a, b]\), i.e.

\[
\mathcal{P}([a, b]) = \{ P = \{x_0, \ldots, x_{N-1}\} : N \in \mathbb{N}, a = x_0 < x_1 < \ldots < x_{N-1} = b \}.
\]

Then we denote by

\[
V^b_a(f) = \sup_{P \in \mathcal{P}([a, b])} \sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)|
\]

the total variation of \(f\).

If \(V^b_a(f) < \infty\) we call \(f\) a function of bounded variation.

We can derive a couple of properties from this definition.

**Corollary 6.6.** Let \(f : [a, b] \to \mathbb{R}\). Then

1. \(f\) is a constant function if and only if \(V^b_a(f) = 0\).
2. \(V^b_a(f) = V^c_a(f) + V^b_c(f)\) for \(a \leq c \leq b\).
3. \(V^b_a(f) = \int_a^b |f'(t)| \, dt\) for continuously differentiable \(f\).

**Proof.**

1. Let \(f\) be constant. Then for each \(x, y \in [a, b]\) we have \(|f(x) - f(y)| = 0\) and therefore \(V^b_a(f) = 0\).

Now let \(V^b_a(f) = 0\). Then for every two points \(x, y \in [a, b]\) we have

\[
|f(x) - f(y)| = 0.
\]

Hence, \(f(x) = f(y)\) and \(f\) is constant on \([a, b]\).

2. Since the triangle inequality holds for real numbers, we have

\[
V^b_a(f) \leq V^c_a(f) + V^b_c(f).
\]

Due to the fact

\[
\{ P = \{x_0, \ldots, x_i, \ldots, x_{N-1}\} : a = x_0 < \ldots < x_i = c < \ldots < x_{N-1} = b \} \subseteq \mathcal{P}([a, b])
\]

we know that \(V^c_a(f) + V^b_c(f) \leq V^b_a(f)\).

3. Let \((P_N)_{N \geq 1}\) be a sequence of partitions of \([a, b]\) with \(|P_N| \to 0\) such that \(V^b_a(f) = \lim_{n \to \infty} \sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)|\). Due to the mean value theorem, for every partition \(P_N\), there exist intermediates points \(\zeta_i \in [x_i, x_{i+1}]\) such that

\[
\sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)| = \sum_{i=0}^{N-1} |f'(\zeta_i)(x_{i+1} - x_i)|.
\]
Obviously

\[
U(P_N) := \sum_{i=0}^{N-1} \inf_{\zeta \in [x_i, x_{i+1}]} |f'(\zeta)(x_{i+1} - x_i)| \\
\leq \sum_{i=0}^{N-1} |f'(\zeta_i)(x_{i+1} - x_i)| \\
\leq \sum_{i=0}^{N-1} \sup_{\zeta \in [x_i, x_{i+1}]} |f'(\zeta)(x_{i+1} - x_i)| =: O(P_N)
\]

holds. Therefore we have \( U(P_N) \leq V_b(f) \leq O(P_N) \).

Since \( f' \) is continuous, we know that \( f' \) is Riemann integrable and the limit of the Riemann sums \( O(P_N) \) and \( U(P_N) \) coincide.

To be able to prove the upper bound of integration errors of functions of bounded variation we have to introduce the Riemann-Stieltjes integral first.

Let \( P_N = (x_0, ..., x_N) \) be a partition of \([a, b]\) and let \( \zeta = \{\zeta_1, ..., \zeta_N\} \) be a set of intermediate values with \( x_{i-1} \leq \zeta_i \leq x_i \) for \( i = 1, ..., N \). Then, for a function \( \phi : [a, b] \to \mathbb{R} \), we denote by

\[
S_\phi(f, P_N, \zeta) = \sum_{i=1}^{N} f(\zeta_i) \cdot (\phi(x_i) - \phi(x_{i-1}))
\]
a Riemann-Stieltjes sum.

Further we define \( |P_N| = \max_{1 \leq i \leq N} (x_i - x_{i-1}) \). If for every set of intermediate values \( \zeta \) the limits

\[
\lim_{N \to \infty, |P_N| \to 0} S_\phi(f, P_N, \zeta)
\]
exist and coincide, we call \( f \) Riemann-Stieltjes integrable in respect to \( \phi \) on \([a, b]\).

The limit is denoted by

\[
\int_a^b f(t) d\phi(t)
\]
and is called the Riemann-Stieltjes integral with respect to \( \phi \). If we choose \( \phi(t) = t \) on \([a, b]\), this definition coincides with the usual Riemann integral.

\begin{lemma} (Partial Integration). Let \( f, \phi : [a, b] \to \mathbb{R} \) and assume that \( f \) is Riemann-Stieltjes integrable with respect to \( \phi \). Then \( \phi \) is also Riemann-Stieltjes integrable with respect to \( f \) and

\[
\int_a^b f(t) d\phi(t) = f(b)\phi(b) - f(a)\phi(a) - \int_a^b \phi(t) df(t)
\]
holds.
\end{lemma}
Proof. Let \( P_N \) be a partition of \( [a, b] \) with a set of intermediate points \( \zeta = (\zeta_1, ..., \zeta_N) \). Let further \( \zeta_0 = a \) and \( \zeta_{N+1} = b \). Then we can also interpret \( \zeta_{N+1} := \{\zeta_0\} \cup \zeta \cup \{\zeta_{N+1}\} \) as a partition of \( [a, b] \) with a set of intermediate points \( P_N \). Further, we obtain

\[
\sum_{i=1}^{N} f(\zeta_i)(\phi(x_i) - \phi(x_{i-1})) = \sum_{i=1}^{N} f(x_i)\phi(\zeta_i) - \sum_{i=1}^{N} f(\zeta_i)\phi(x_{i-1}) \\
= \sum_{i=1}^{N-1} \phi(x_i)(f(\zeta_i) - f(\zeta_{i+1})) + f(\zeta_N)\phi(x_N) - f(\zeta_1)\phi(x_0) \\
= \sum_{i=0}^{N} \phi(x_i)(f(\zeta_i) - f(\zeta_{i+1})) + f(\zeta_{N+1})\phi(x_N) - f(\zeta_0)\phi(x_0) \\
= f(b)\phi(b) - f(a)\phi(a) - \sum_{i=0}^{N} \phi(x_i)(f(\zeta_{i+1}) - f(\zeta_i)).
\]

Since \( f \) is Riemann-Stieltjes integrable with respect to \( \phi \), the choice of intermediate points does not matter and we get

\[
\lim_{N \to \infty, |P_N| \to 0} \sum_{i=0}^{N} \phi(x_i)(f(\zeta_{i+1}) - f(\zeta_i)) = f(b)\phi(b) - f(a)\phi(a) - \int_{a}^{b} f(t)d\phi(t)
\]

for every partition \( \zeta_{N+1} \) of \( [a, b] \) with a set of intermediate points \( P_N \). Therefore, by definition, \( \phi \) is Riemann-Stieltjes integrable with respect to \( f \) and

\[
\int_{a}^{b} \phi(t)df(t) = f(b)\phi(b) - f(a)\phi(a) - \int_{a}^{b} f(t)d\phi(t)
\]

holds. \( \square \)

Lemma 6.8. Let \( f, \phi : [a, b] \to \mathbb{R} \) be functions of bounded variation. Then

\[
\left| \int_{a}^{b} f(t)d\phi(t) \right| \leq \sup_{a \leq t \leq b} |f(t)| \cdot V_a^b(\phi).
\]

Proof. Let \( P_N = (x_0, ..., x_N) \) be a partition of \( [a, b] \) and \( \zeta = (\zeta_1, ..., \zeta_N) \) with \( x_{i-1} \leq \zeta_i \leq x_i \) for \( i = 1, ..., N \). Then we have

\[
\left| \sum_{i=1}^{N} f(\zeta_i) \cdot (\phi(x_i) - \phi(x_{i-1})) \right| \leq \sum_{i=1}^{N} |f(\zeta_i)| \cdot |(\phi(x_i) - \phi(x_{i-1}))| \\
\leq \sup_{a \leq t \leq b} |f(t)| \cdot \sum_{i=1}^{N} |\phi(x_i) - \phi(x_{i-1})| \\
\leq \sup_{a \leq t \leq b} |f(t)| \cdot V_a^b(\phi).
\]

\( \square \)
With these tools we can prove the following upper bound for QMC algorithms applied to functions of bounded variation in one dimension.

**Theorem 6.9** (Koksma-Hlawka inequality). Let \( f : [0, 1] \to \mathbb{R} \) be a function of bounded variation and let \( P = \{x_1, ..., x_N\} \) be a set of points in \([0, 1]\). Then we have

\[
e(Q_P, f) \leq V_{HK}(f) \cdot D^*_N(P).
\]

**Proof.** Let \( x_0 = 0 \leq x_1 \leq ... \leq x_N \leq x_{N+1} = 1 \). First we show the equality

\[
\frac{1}{N} \sum_{i=1}^{N} f(x_i) - \frac{1}{N} \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (t - \frac{i}{N})df(t).
\]

By using partial integration of the Riemann-Stieltjes integral we have

\[
\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (t - \frac{i}{N})df(t) = \int_{0}^{1} tdf(t) - \sum_{i=0}^{N} \frac{i}{N} (f(x_{i+1}) - f(x_i))
\]

\[
= 1 \cdot f(1) - 0 \cdot f(0) - \int_{0}^{1} f(t)dt + \frac{1}{N} \sum_{i=1}^{N} f(x_i) - f(x_{n+1})
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} f(x_i) - \int_{0}^{1} f(t)dt.
\]

The last step holds because \( x_{N+1} = 1 \).

By using (6.1) and Lemma 6.8 we have

\[
\left| \int_{0}^{1} f(t)dt - \frac{1}{N} \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} (t - \frac{i}{N})df(t) \right| \leq \sum_{i=0}^{N} \sup_{x_i \leq t \leq x_{i+1}} \left| t - \frac{i}{N} \right| \cdot V^{x_{i+1}}_{x_i}(f).
\]

One can show that

\[
\left| t - \frac{i}{N} \right| \leq D^*_N(P)
\]

for \( i \in \{0, ..., N\} \), see for example [9, Remark 2.29]. Then we have by Corollary 6.6

\[
e(Q_P, f) = \left| \int_{0}^{1} f(t)dt - \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right| \leq D^*_N(P) \cdot \sum_{i=0}^{N} V^{x_{i+1}}_{x_i}(f) = D^*_N(P) \cdot V^1_0(f).
\]
6.3 Functions of bounded variation in general dimensions

In this section we give a definition of bounded variation in general dimensions and use that to generalize the Koksma-Hlawka inequality. For higher dimensions there are several possibilities to generalize the variation of a function. Later we will use the variation in the sense of Hardy and Krause.

We define the operator \( \triangle_{h_k} \) for \( h_k \in \mathbb{R}, k \in \{1, \ldots, d\} \), as
\[
\triangle_{h_k}(f, x) = f(x_1, \ldots, x_{k-1}, x_k + h_k, x_{k+1}, \ldots, x_d) - f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_d).
\]

Further we define the operator \( \triangle_{h_1, \ldots, h_k} \) recursively as
\[
\triangle_{h_1, \ldots, h_k}(f, x) = \triangle_{h_k}(\triangle_{h_1, \ldots, h_{k-1}}(f, x)).
\]

**Definition 6.10.** Let \( f : [a, b] \subset \mathbb{R}^d \to \mathbb{R} \), let \( \pi_k \) be a partition
\[
a_k = t_{k1} < \ldots < t_{kN_k+1} = b_k
\]
of \([a, b] \) and let \( h_k^i = t_k^{i+1} - t_k^i \) for \( k = 1, \ldots, d \). Then we denote by
\[
V^{(d)}_{\text{V}}(f) = \sup_{(\pi_1, \ldots, \pi_d)} \sum_{k=1}^{N_k} \sum_{i_k=1}^{N_k} |\triangle_{h_1^i, \ldots, h_k^i}(f, (x_1^i, \ldots, x_k^i))|
\]
the Vitali variation of \( f \).

If a function \( f : [a, b] \to \mathbb{R} \) is constant in one variable, we have \( V^{(d)}_{\text{V}}(f) = 0 \). We want to avoid this consequence by the following interpretation of bounded variation.

**Definition 6.11.** Let \( f : [a, b] \subset \mathbb{R}^d \to \mathbb{R} \) and let \( \alpha, \tilde{\alpha} \) be a pair of ordered subsets which gives a partition of \( \{1, \ldots, d\} \) with \( |\alpha| = s \). For each such pair and for each
\[
y = (y_1, \ldots, y_d) \in [a_{\alpha_1}, b_{\alpha_1}] \times \ldots \times [a_{\alpha_s}, b_{\alpha_s}]
\]
we denote by \( f^j_{\alpha} \) the function of \( n-s \) variables \( z_1, \ldots, z_{n-s} \) given by \( f(x_1, \ldots, x_d) \), where \( x_{\alpha_i} = y_i \) and \( x_{\tilde{\alpha}_j} = z_j \). Then we denote by
\[
V_{\text{HK}}(f) = \sup_{\alpha} \sup_y \sum_{j=1}^{d} V^{(j)}_{\text{V}}(f_{\alpha}^j)
\]
the Hardy-Krause variation of \( f \).

Further we call \( f \) a function of bounded Hardy-Krause variation if
\[
V_{\text{HK}}(f) < \infty.
\]

For functions \( f \) with continuous mixed partial derivatives up to order 1 there is a connection between the bounded variation in the sense of Hardy and Krause and the norm \( ||f||_{d,1} \) given by
\[
V_{\text{HK}}(f) = ||f||_{d,1} - |f(1, \ldots, 1)|.
\]

Therefore it is reasonable that the Koksma-Hlawka inequality holds for functions of bounded Hardy-Krause variation, which is indeed the case.
Theorem 6.12 (General Koksma-Hlawka inequality). Let $f : [0,1]^d \to \mathbb{R}$ be a function of bounded Hardy-Krause variation and let $P = \{x_0, ..., x_{N-1}\}$ be a set of points in $[0,1]^d$. Then we have

$$e(Q_P,f) \leq V_{HK}(f) \cdot D_N^*(P).$$

Proof. See [7, Theorem 5.5]

By using the General Koksma-Hlawka inequality and Corollary 4.9 we can give an upper error bound for QMC algorithms applied to functions with bounded variation.

Corollary 6.13. Let $L \subset \mathbb{R}^d$ be an admissible lattice with admissibility constant $k$ and let $f : [0,1]^d \to \mathbb{R}$ be a function of bounded Hardy-Krause variation. Then there exists a constant $c(k, \det L) \in \mathbb{R}$ such that for every $T = (t_1, ..., t_d) \in \mathbb{R}^d$ with no vanishing component and $x \in \mathbb{R}^d$ for point sets $P_{T,x} = T^{-1} \cdot (L - x) \cap [0,1]^d$ with $N$ elements the bound

$$e(Q_{P_{T,x}}, f) \leq c \cdot V_{HK}(f) \cdot \frac{(\ln N)^{d-1}}{N}$$

holds.
6.4 Sobolev spaces

In this section we introduce Sobolev spaces and their connection to the already introduced function spaces. Afterwards we give also an upper bound for the error of algorithms applied to Sobolev space functions.

Definition 6.14. Let $A \subseteq \mathbb{R}^d$ be open and let $f : \bar{A} \to \mathbb{R}$ be locally integrable. Then for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ we call a locally integrable function $g : \bar{A} \to \mathbb{R}$ the $\alpha$-th weak derivative, if for every $\phi \in C_c^\infty(\bar{A})$ the equation

$$\int_A f(x)D_\alpha \phi(x)dx = (-1)^{|\alpha|} \int_A g(x)\phi(x)dx,$$

where $D_\alpha \phi = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \phi$, holds.

In the upcoming pages we refer to the $\alpha$-th weak derivative of a function $f$ by $D_\alpha f$.

Definition 6.15. Let $A \subseteq \mathbb{R}^d$ be open, let $1 \leq q < \infty$ and $k \in \mathbb{N}$. The Sobolev space with dominating mixed smoothness $W^{k,q}(\bar{A})$ contains of all locally integrable functions $f : \bar{A} \to \mathbb{R}$ such that $f$ has weak derivatives for all $\alpha \in \mathbb{N}_0^d$ with $||\alpha||_\infty \leq k$ and such that $f$ and $D_\alpha f$ for $||\alpha||_\infty \leq k$ are in $L^q(A)$.

We equip $W^{k,q}(\bar{A})$ with the norm

$$||f||_{k,q,\text{mix}} = \left(\sum ||D_\alpha f||_{L^q(A)}^q\right)^{1/q}.$$

Further we define the Sobolev space with zero trace as

$$W^{k,q}_0(\bar{A}) = \{f \in W^{k,q}(\bar{A}) : \text{supp}(f) \subset A\}.$$

The Sobolev spaces with trace zero are indeed well defined as the trace operator and its properties show. For more details see for example [8, Chapter 6].

Example 6.16. The function

$$f(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}], \\ 1/2 & \text{otherwise} \end{cases}$$

is an element of $W^{1,2}([0, 1])$. The weak derivative of $f$ is

$$f'(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}], \\ 0 & \text{otherwise} \end{cases}$$

and we have

$$||f||_{1,2,\text{mix}}^2 = \int_0^1 f(x)^2 dx + \int_0^1 f'(x)^2 dx = \frac{x^3}{3}\bigg|_0^{1/2} + \frac{x^4}{4}\bigg|_0^{1/2} + \frac{x^{1/2}}{2}\bigg|_0^1 = \frac{2}{3} < \infty.$$
However, the function

\[ g(x) = \begin{cases} 
  x & \text{if } x \in [0, \frac{1}{2}], \\
  1 & \text{otherwise}
\end{cases} \]

is not in \( W^{1,2}([0,1]) \), since there exists no weak derivative of \( g \). If there would exist a function \( h \in L^2([0,1]) \) such that

\[
\int_0^1 g(x)\phi'(x)dx = -\int_0^1 h(x)\phi(x)dx
\]

for all \( \phi \in C^\infty_c([0,1]) \), we would have

\[
-\int_0^1 h(x)\phi(x)dx = \int_0^{1/2} x\phi'(x)dx + \int_{1/2}^1 \phi'(x)dx
\]

\[
= x\phi(x)\bigg|_0^{1/2} - \int_0^{1/2} \phi(x)dx + \phi(x)\bigg|_{1/2}^1
\]

\[
= -\phi(1/2) - \int_0^{1/2} \phi(x)dx.
\]

Now let us choose a sequence of smooth functions \( \phi_n \) with

\[
0 \leq \phi_n \leq 1, \quad \phi_n(1/2) = 1 \quad \text{and} \quad \lim_{n \to \infty} \phi_n(x) = 0 \quad \text{for all } x \neq \frac{1}{2}
\]

for \( n \in \mathbb{N} \). Then we have

\[ 1 = \phi_n(1/2) = 2 \cdot \left( \int_0^1 h(x)\phi_n(x)dx - \int_0^{1/2} \phi_n(x)dx \right) \]

and, by taking the limit \( n \to \infty \) on both sides, the wrong proposition

\[ 1 = 2 \cdot \lim_{n \to \infty} \left( \int_0^1 h(x)\phi_n(x)dx - \int_0^{1/2} \phi_n(x)dx \right) = 0. \]
Remark 6.17. In this example we have seen a non-continuous functions with no weak derivative. In fact, continuity is a necessary property for the existence of weak derivatives. Vice versa, it is sufficient for the existence of a first order weak derivative, that the function is absolutely continuous.

For admissible lattices $L$ and $T \in (\mathbb{R} \setminus \{0\})^d$ Skriganov has shown upper error bounds for the error of algorithms

$$Q_T(f) = \det L \cdot \sum_{z \in T^{-1} \cdot L \cap [0,1]^d} f(z)$$

applied to functions $f$, which are an element of a Sobolev space with zero trace, in dependence on the smoothness of $f$ as the following statement shows.

Theorem 6.18. Let $L \subset \mathbb{R}^d$ be an admissible lattice with admissibility constant $r$, let $f \in W_0^{k,q}([0,1]^d)$ and let $1 < q < \infty$. Then there exist constants $c(r, \det L), c_q(r, \det L)$ such that for $T \in \mathbb{R}^d$ with no vanishing component the bounds

$$e(Q_T, f) \leq c \cdot ||f||_{k,1,mix} \cdot \frac{(\ln (\prod_{j=1}^d |t_j|)^{d-1})}{\prod_{j=1}^d |t_j|^k}$$

and

$$e(Q_T, f) \leq c_q \cdot ||f||_{k,q,mix} \cdot \frac{(\ln (\prod_{j=1}^d |t_j|)^{(d-1)/2})}{\prod_{j=1}^d |t_j|^k}$$

hold for $\prod_{j=1}^d |t_j| \geq 2$.

Proof. This proof is similarly technical as the proof of Skriganov’s theorem and can be found for example in [13, Theorem 2.1].

If we apply the previous theorem on the sequence of admissible lattices defined in section 5.3 with a Frolov matrix $B$, we get for the algorithms

$$Q_n(f) = \frac{\det B}{n} \cdot \sum_{z \in \mathbb{L}_n \cap [0,1]^d} f(z)$$

the following result:

Figure 10: The function $g$, which does not have a weak derivative.
Corollary 6.19. Let $1 < q < \infty$ and let $d \in \mathbb{N}$. Then there exist constants $c, c_q \in \mathbb{R}$ such that for the sequence of algorithms $(Q_n)_{n \in \mathbb{N}}$

$$e(Q_n, W^{k,1}_0([0,1]^d)) \leq c \cdot \frac{(\ln n)^{d-1}}{n^k}$$

and

$$e(Q_n, W^{k,q}_0([0,1]^d)) \leq c_q \cdot \frac{(\ln n)^{(d-1)/2}}{n^k}$$

hold.

Proof. Let $L_1$ be generated by a Frolov matrix $B$ with positive determinant. We obtain that the algorithms $Q_n$ are a particularly chosen case of the algorithms $Q_T$ by choosing

$$T = \left( \frac{n^{1/d}}{\det B^{1/d}}, \ldots, \frac{n^{1/d}}{\det B^{1/d}} \right)$$

and

$$L = \mathcal{L}_1^\perp.$$

Then the result follows by Theorem 6.18. \hfill \Box

Standard techniques, which can be found for example in [16, Theorem 1.1], extend the previous corollary to general Sobolev spaces, i.e. spaces without boundary conditions.

Corollary 6.20. Let $1 < q < \infty$ and let $d \in \mathbb{N}$. Then there exist constants $c, c_q \in \mathbb{R}$ such that for the sequence of algorithms $(Q_n)_{n \in \mathbb{N}}$

$$e(Q_n, W^{k,1}_0([0,1]^d)) \leq c \cdot \frac{(\ln n)^{d-1}}{n^k}$$

and

$$e(Q_n, W^{k,q}_0([0,1]^d)) \leq c_q \cdot \frac{(\ln n)^{(d-1)/2}}{n^k}$$

hold.
In this thesis we have shown how to construct a sequence of algorithms \((Q_n)_{n \in \mathbb{N}}\), which approximates the integral of specific functions \(f\) over the unit cube as integration domain. We have used a Frolov matrix \(B\), a Vandermonde matrix with conjugate numbers over \(\mathbb{Q}\) as entries, as a generator of an admissible lattice \(\mathcal{L}\) and obtained a sequence of admissible lattices \(\mathcal{L}_n\) by shrinking \(\mathcal{L}\) by the factor \(\frac{n^{1/d}}{\det B^{1/d}}\) in each of the \(d\) dimensions. To show the admissibility of \(\mathcal{L}_n\) was an essential part of this thesis. The sequence of algorithms we were looking for can now be given by

\[
Q_n(f) = \frac{\det B}{n} \cdot \sum_{z \in \mathcal{L}_n \cap [0,1]^d} f(z).
\]

We have covered the cases that \(f\) is continuously differentiable, has bounded variation, or is an element of a Sobolev space. There are several other possibilities where the given method works and gives nice error bounds of \(Q_n\). For instance, the case of Besov-Triebel-Lizorkin spaces, which can be found for example in [18].
Plots

Left: The polynomial $P_d$; Right: The Chebyshev-polynomial $Q_d$:

For dimension $d = 4$:

$d = 6$:

$d = 10$:
References


