Dimensions and bases of $C^1$-smooth isogeometric functions on three-dimensional two-patch domains

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Abstract

We analyze the spaces of trivariate $C^1$–smooth isogeometric functions on two-patch domains. Our aim is to generalize the corresponding results from the bivariate [15, 30, 36] to the trivariate case.

In the first part of the present work, we introduce the notion of gluing data and use it to define glued spline functions on two-patch domains. Applying the fundamental observation that “matched $G^k$–constructions always yield $C^k$–continuous isogeometric elements”, see [23], to graph hypersurfaces in four-dimensional space allows us to characterize $C^1$–smooth geometrically continuous isogeometric functions as the push-forwards of these functions for suitable gluing data.

The second part is devoted to various special classes of gluing data. We analyze how the generic dimensions depend on the number of knot spans (elements) and on the spline degree and we show how to construct locally supported basis functions in specific situations. The basis construction relies on the fact that we use rational arithmetic. By doing so, we identify coefficient patterns, which in the following allow us to revise the basis computation and present a more efficient construction procedure.
We then perform numerical experiments with $L^2$-projection in order to explore the approximation power of the resulting spline functions for trilinearly parameterized domains. Despite the existence of locally supported bases, we observe a reduction of the approximation order for low degrees, and we provide a theoretical explanation for this locking.

The last part is devoted to trilinearly parameterized two-patch domains. We develop the theoretical framework to explore the $C^1$-smooth isogeometric space. Finally we prove the numerically obtained dimension result from the previous part in this specific situation and describe a simple explicit basis construction which consists of locally supported functions.
Zusammenfassung


Im ersten Teil dieser Arbeit führen wir den Begriff der Übergangsdaten ein und nutzen diese, um zusammengesetzte Splinefunktionen auf Zwei-Patch-Geometrien zu definieren. Im Weiteren verwenden wir die wesentliche Aussage, dass zusammenpassende $G^k$-glatte Konstruktionen immer $C^k$-stetige isogeometrische Elemente ergeben, siehe [23]. Angewandt auf vier-dimensionale Graph-Hyperflächen führt uns das zur Darstellung von $C^1$-glatten geometrisch stetigen isogeometrischen Funktionen mit Hilfe dieser zusammengesetzten Funktionen für passende Übergangsdaten.

Anschließend untersuchen wir die Approximationseigenschaften der sich ergeben-
den Splinefunktionen für den Fall der trilinearen Parametrisierungen anhand von
numerischen Beispielen mit $L^2$–Projektionen. Obwohl sich eine Basis bestehend aus
Funktionen mit lokalem Träger erstellen lässt, sehen wir eine Reduktion der Approx-
imationsordnung für geringe Splinegrade. Wir geben eine theoretische Erklärung für
diese Unstimmigkeit an.

Im letzten Teil der Arbeit konzentrieren wir uns auf trilineare Zwei-Patch-Geometrien.
Für diese untersuchen wir die Theorie des Raums der $C^1$–glatten Splinefunktionen
ausführlicher. Mit diesen Resultaten können wir die Dimensionsformel, welche zu-
vor nur durch numerische Experimente erhalten wurde, für diesen Fall beweisen und
einfache explizite Formeln für die Basisfunktionen angeben.
# Contents

1 Introduction

1.1 Smooth multi-patch domains ........................................ 4  
1.2 Geometric continuity .............................................. 5  
  1.2.1 $G^r$–continuity for $d = 1$ .................................. 6  
  1.2.2 $G^1$–continuity for $d = 2$ .................................. 6  
  1.2.3 $G^2$–continuity for $d = 2$ .................................. 10  
  1.2.4 $G^1$–continuity for $d = 3$ .................................. 12  
1.3 Outline ............................................................... 12  

2 Characterization of $C^1$-smooth isogeometric functions .... 15  
  2.1 Geometrically continuous isogeometric functions .......... 19  
  2.2 Construction of non-trivial spaces $V_F$ ....................... 23  
  2.3 Types of gluing data ............................................... 26  

3 Dimension of the space $V_F$ ........................................ 31  
  3.1 The generic dimension ............................................ 31  
  3.2 Dimensions for different types of gluing data ................ 34
4 Basis functions

4.1 Existence of a locally supported basis 43

4.2 Basis functions for trilinear geometric gluing data 46

4.3 Basis functions for linear gluing data 51

5 Approximation power of trilinear geometric gluing data 53

6 Trilinear two-patch domains

6.1 $C^1$–smooth isogeometric functions 60

6.2 The generic case 67

6.3 Dimension and basis of the space $V_F^\Gamma$ 74

7 Conclusion and future work 83

List of Figures 85

List of Tables 88

Bibliography 90
Chapter 1

Introduction

Isogeometric Analysis (IGA) closes the gap between mathematical concepts used in Computer Aided Design (CAD) and Finite Element Methods [27]. The main idea of IGA is to use the representation of CAD models, typically parameterizations by piecewise parametric Bézier- or B-spline functions or their extension to NURBS – non-uniform rational B-splines [26, 49], also in the analysis of partial differential equations (PDEs). Moreover, IGA allows for discretization spaces of high order smoothness, which is especially beneficial when solving high order PDEs [6, 16].

Fourth order PDEs require discretization spaces that are $C^1$–smooth, since their weak formulations involve second derivatives. Examples of such problems include the Cahn-Hilliard equation [17, 21], the Navier-Stokes-Korteweg equation [22, 56], Kirchhoff-Love shells [7, 37] and the biharmonic equation [3, 56].

Obviously, spline based representations of the physical domain, as used in CAD models, lead to high order smoothness within a single patch, which is topologically equivalent to a box. The construction of smooth multi-patch domains, which are needed to describe complex domains, is non-trivial and the task of ongoing research.
1.1 Smooth multi-patch domains

There are different strategies to obtain continuity across the interfaces of neighboring patches. One promising approach is to couple multi-patch parameterizations across these interfaces. These coupling constraints can be enforced either weakly or strongly.

The weak enforcement of these constraints can be done using Nitsche’s method as in [1, 45]. Another possibility are Langrangian multipliers, they are used within the isogeometric mortar methods [12] and the IETI (Isogeometric Tearing and Interconnecting) solvers [25, 39].

The strong imposition of coupling constraints is especially well understood for the construction of $C^0$–smooth isogeometric multi-patch domains, see [6]. B-splines with enhanced smoothness on multi-patch domains were obtained in [13] by suitably merging or modifying basis functions locally in the vicinity of the boundary of each patch. The potential of isogeometric spline forests as a basis in IGA was discussed in [52]. Similarly, T-splines, which generalize NURBS to allow T-junctions [54], only admit $C^0$–smooth functions. The approximation properties of T-splines on two-patch domains were discussed in [5].

To obtain high order smoothness (especially $C^1$–continuity) across the interfaces of a multi-patch domain using T-splines one needs special constructions around extraordinary vertices [4, 5]. Likewise, subdivision surfaces [48], which are popular especially in Computer Graphics, can be constructed with high smoothness across patch boundaries except around extraordinary vertices. Subdivision algorithms allow to efficiently construct free-form surfaces of arbitrary topology, as for example the Catmull-Clark subdivision [14]. Although, these constructions have optimal convergence power for special configurations [2, 43], this is not true in general and special attention needs to be paid on the numerical quadrature schemes [28].
Geometric continuity

Complex CAD models of arbitrary topology can also be treated using trimming techniques [38]. Conventional trimmed-NURBS representations often lead to gaps when gluing together trimmed patches [41, 53] and advanced techniques are required when coupling matching and non-matching discretizations [50].

Another well-know approach to construct smooth multi-patch domains is to use geometric continuity [19, 46, 47], which will be discussed in more detail in the following sections.

1.2 Geometric continuity

Consider two adjacent $C^r$–smooth patches

$$p, q : \hat{\Omega} = [0, 1]^d \to \mathbb{R}^m, \quad m \geq d,$$

which meet at a common regular boundary

$$p(\cdot, 0) = q(\cdot, 0).$$

If we require the two patches to meet $C^r$–continuous at the common interface, they would need to agree up to the $r$-th derivative at the junction. In contrast, geometric continuity of $r$-th order ($G^r$-continuity) has only requirements on the shape of the two patches in the vicinity of the shared boundary.

For details on the following definition see [26, 47].

**Definition 1.1.** We say that two domains join $G^r$–smoothly at a common boundary, if there exists a reparameterization (parameter transformation) which transforms them to join with $C^r$–smoothness.
1.2.1 $G^r$–continuity for $d = 1$

For simplicity, let us consider the easiest case, which is that two $C^r$–smooth curves $p(u)$ and $q(v)$ meet in a common point $p(0) = q(0)$. They join $G^r$–smooth, if their $r$-th derivatives agree after a suitable reparameterization.

We can reparameterize the curve $q(v)$ using

$$v \mapsto v(t), \quad v(0) = 0.$$

Note that we need to use an orientation-preserving reparameterization. We then obtain that the curves meet with $G^r$–continuity at the common point if

$$\frac{d^r}{dt^r} p(-t)|_{t=0} = \frac{d^r}{dt^r} q(v(t))|_{t=0}.$$

1.2.2 $G^1$–continuity for $d = 2$

Geometric continuity in higher dimensions is a more involved problem. The exploration of $C^1$–smoothness on planar multi-patch surfaces, including the case of surfaces which possess extraordinary vertices, using the concept of geometric continuity, is the task of many recent publications.

First results on multi-patch parameterizations of a disk were presented in [43]. The numerical experiments indicated a reduction of the approximation power, if extraordinary vertices were present.

The main starting point of the majority of existing literature is the important observation that $C^r$–smoothness of an isogeometric function is based on the $G^r$–continuity of the associated parameterization [23]. Recently, this result allowed to construct
globally $C^1$–smooth multi-patch spline spaces, e.g. [8].

For simplicity, let us again consider a two-patch configuration (the existence of extraordinary vertices does not affect the approach). Assume that we have given two quadrilateral planar patches $\Omega^{(1)}, \Omega^{(2)}$, both described by regular parameterizations

$$F^{(i)} : [0,1]^2 = \hat{\Omega} \rightarrow \Omega^{(i)}, \quad i \in \{1,2\}. \tag{1.1}$$

Let $S^p_{r,k}$ denote the space of spline functions on $[0,1]$ of degree $p$ and regularity $1 \leq r \leq p - 1$ (at the inner knots) with respect to the open knot vector

$$T^p_{r,k} = (0,0,\ldots,0, \tau_1, \tau_1,\ldots,\tau_1, \tau_k, \tau_k,\ldots,\tau_k,1,1,\ldots,1), \tag{1.1}$$

where $0 < \tau_i < \tau_{i+1} < 1$ for all $1 \leq i \leq k - 1$ and $k$ is the number of different inner knots. Note that the space $S^p_{r,k}$ contains spline functions which are at least $C^r$–smooth. The two subdomains are described by parametric representations $F^{(1)}$ and $F^{(2)}$ with coordinate functions from the tensor-product spline space

$$\mathcal{P} = S^p_{1,k} \otimes S^p_{1,k},$$

for certain values of the degree $p$ and the knot number $k$. These parameterizations, which are defined on two copies of the domain $[0,1]^2$, form the two-patch geometry mapping

$$F = (F^{(1)}, F^{(2)}) \in \mathcal{P}^2 \times \mathcal{P}^2.$$ 

Furthermore, we assume that the two patches share a common face, parameterized by the two copies of the face $\hat{\Gamma} = [0,1] \times \{0\}$, this means

$$F^{(1)}(u,0) = F^{(2)}(u,0), \quad u \in [0,1].$$
The goal is to construct globally $C^1$–smooth isogeometric functions on $\Omega$, hence we are interested in the space

$$\mathcal{V}_1 = [(\mathcal{P} \times \mathcal{P}) \circ F^{-1}] \cap C^1(\Omega^{(1)} \cup \Omega^{(2)}).$$

In detail, we consider functions $\phi \in (\mathcal{P} \times \mathcal{P}) \circ F^{-1}$. For any such function $\phi$ there exist two spline functions $g^{(1)}, g^{(2)} \in \mathcal{P}$ such that

$$(\phi|_{\Omega^{(i)}})(\xi) = g^{(i)}(\xi) = (g^{(i)} \circ (F^{(i)})^{-1})(\xi), \quad \xi = (\xi_1, \xi_2) \in \Omega^{(i)}, \ i \in \{1, 2\},$$

or equivalently

$$(\phi|_{\Omega^{(i)}})(F^{(i)}(x)) = g^{(i)}(F^{(i)}(x)) = g^{(i)}(x), \quad x = (u, v) \in \hat{\Omega} = [0, 1]^2, \ i \in \{1, 2\}.$$

As an additional smoothness property we require that the first derivatives of $\phi^{(1)}$ and $\phi^{(2)}$ take identical values at the interface, since $\phi \in C^1(\Omega^{(1)} \cup \Omega^{(2)})$.

We consider the associated graph surfaces in three-dimensional space $\Sigma$ of $\phi$, which consists of two patches

$$\Sigma^{(i)} = (F^{(i)}, g^{(i)})^T : \hat{\Omega} \to \mathbb{R}^3, \ i \in \{1, 2\}.$$

The function $\phi$ is $C^1$–smooth if and only if the two patches have identical tangent planes along the interface

$$\Sigma^{(1)}(u, 0) = \Sigma^{(2)}(u, 0), \quad u \in [0, 1]. \quad (1.2)$$
This can similarly be expressed by the following two equations

\[ g^{(1)}(u, 0) = g^{(2)}(u, 0), \]  
\[ \alpha^{(1)}(u) \partial_u F^{(2)}|_{v=0} - \alpha^{(2)}(u) \partial_u F^{(1)}|_{v=0} + \beta(u) \partial_v F^{(1)}|_{v=0} = 0, \]

see [26, 47]. The partial derivative operator \( \partial_u \) indicates the differentiation with respect to the \( u \)-direction and similarly for \( \partial_v \) and the \( v \)-direction.

Note that the three factors \( \alpha^{(1)}, \alpha^{(2)} \) and \( \beta \) are uniquely defined up to a common factor \( \lambda \in \mathbb{R}_{>0} \)

\[ \alpha^{(1)}(u) = \lambda \det \left( \partial_u F^{(1)}|_{v=0}, \partial_v F^{(1)}|_{v=0} \right), \]
\[ \alpha^{(2)}(u) = \lambda \det \left( \partial_u F^{(2)}|_{v=0}, \partial_v F^{(2)}|_{v=0} \right), \]
\[ \beta(u) = \lambda \det \left( \partial_u F^{(2)}|_{v=0}, \partial_u F^{(1)}|_{v=0} \right). \]

In [36] and [29] the authors used equations (1.3) and (1.4) to investigate the space of \( C^1 \)-smooth isogeometric functions on bilinearly parameterized planar two- and multi-patch domains, respectively. The use of this approach allowed to have \( C^1 \)-smoothness also in the vicinity of extraordinary vertices and in both papers full approximation order was obtained in numerical experiments.

This concept was first extended to more general parameterized domains in [15]. The so-called analysis-suitable \( G^1 \) parameterizations allow to construct bivariate \( C^1 \)-smooth isogeometric function spaces with optimal approximation properties. The specific case of two-patch domains was considered in [30] and the generalization to multi-patch domains in [31].
A similar approach based on the concept of geometric continuity is also used in [42] to define the space of $G^1$ spline functions on surfaces of arbitrary topology, where beside rectangular patches, also triangular ones are allowed.

The dimension of the space of $C^1$–smooth isogeometric functions in all of the above mentioned cases depends on the domains parameterization, see [30, 42]. This is overcome in [32] by constructing a suitable subspace, the Argyris isogeometric space. It has a simpler structure than the full $C^1$ space, while maintaining optimal approximation power.

The extension from planar surfaces to surfaces in space is rather straightforward. The space of $C^1$–smooth splines on analysis-suitable $G^1$ multi-patch geometries was theoretically investigated in [15]. Numerical experiments showed that also in this case full approximation power is obtained [15, 31].

1.2.3 $G^2$–continuity for $d = 2$

The concept of geometric continuity can also be used to construct the space of $C^2$–smooth isogeometric splines on bilinearly parameterized multi-patch domains. Let us consider the two-patch geometry

\[ \mathbf{F} = (\mathbf{F}^{(1)}, \mathbf{F}^{(2)}) \in \tilde{\mathbf{P}}^2 \times \tilde{\mathbf{P}}^2, \]

with

\[ \tilde{\mathbf{P}} = S^p_{2,k} \otimes S^p_{2,k}. \]

Then we are interested in the space

\[ \mathcal{V}^2 = [((\mathbf{P} \times \mathbf{P}) \circ \mathbf{F}^{-1}) \cap C^2(\Omega^{(1)} \cup \Omega^{(2)})) \]

of globally $C^2$–continuous isogeometric functions on $\Omega$.

Using the result from [23], we know that an isogeometric function $\phi$ is an element of $V^2$ if and only if the two patches of the associated graph surface $\Sigma$ meet with $G^2$–smoothness along the common interface (1.2).

Therefore, besides satisfying equations (1.3) and (1.4) to guarantee $G^1$–continuity, the two patches $\Sigma^{(1)}$ and $\Sigma^{(2)}$ need to fulfill the $G^2$ condition (cf. [26, 47])

$$\det \left( W(u), \partial_u \Sigma^{(1)}|_{v=0}, \partial_v \Sigma^{(1)}|_{v=0} \right) = 0, \quad u \in [0,1], \quad (1.5)$$

with

$$W(u) := (W_1(u), W_2(u), W_3(u))^T := \alpha^{(2)}(u) \partial^2 F^{(2)}|_{v=0}$$

$$- \left( \alpha^{(2)}(u) \partial^2 F^{(1)}|_{v=0} + 2\beta(u)\alpha^{(2)}(u) \partial_u \partial_v F^{(1)}|_{v=0} \right) + \beta^2(u) \partial^2 F^{(1)}|_{v=0} \right). \quad (1.6)$$

The equations (1.5) and (1.6) combined with (1.3) and (1.4) can be used to analyze the space of $C^2$–smooth isogeometric functions on the domain $\Omega$. Dimension results and a basis construction for bilinearly parameterized two- and general multi-patch domains are provided in [34] and [33], respectively. In both cases the authors were able to show optimal convergence rates using numerical experiments.

The generalization to bilinear-like $G^2$ two patch geometries was considered in [35], again numerical experiments indicate optimal approximation power.
1.2.4 \( G^1 \)-continuity for \( d = 3 \)

To the best of our knowledge, there exist very few results on the space of geometrically continuous splines using hexahedral meshes. Tricubic splines on unstructured meshes were considered in [57]. These splines are constructed using a blending method, which yields functions that are only \( C^0 \)-smooth around extraordinary vertices, \( C^1 \)-continuous across interfaces between irregular and regular regions and \( C^2 \)-smooth in the interior of regular regions.

Bivariate \( C^1 \)-smooth spline spaces were used to construct trivariate ones using a sweeping approach in [44], which is very restrictive on the available topologies. \( C^1 \)-smooth isogeometric spaces can also be constructed using tetrahedral meshes, see e.g.[18, 24, 40, 51, 55, 58] and the references therein.

In this thesis we generalize the results of Section 1.2.2 to the trivariate case. The main results have been published in [9, 10, 11].

1.3 Outline

In the first part of the present work, we establish the notion of gluing data and use it to define glued spline functions on two-patch domains. Employing the fact that “matched \( G^k \)-constructions always yield \( C^k \)-continuous isogeometric elements” [23] allows us to characterize \( C^1 \)-smooth geometrically continuous isogeometric functions as the push-forwards of these functions for suitable gluing data. Furthermore, we introduce various special classes of gluing data.
In Chapter 3, we analyze how the generic dimensions depend on the number of knot spans (elements) and on the spline degree. We then show how to construct locally supported basis functions in specific situations in Chapter 4. Based on these preparations we present coefficient patterns of two types of gluing data, which were found to be promising for good approximation power, in further detail. These patterns allow us to efficiently compute a basis of the space of $C^1$–smooth spline functions. We exploit this fact in Chapter 5 where we numerically analyze the approximation power of the bases in case of trilinear geometric gluing data via $L^2$–fitting.

Finally, in the last part of the thesis, we develop the theoretical framework to investigate the $C^1$–smooth isogeometric spline space in case of trilinear geometric gluing data. This allows us to prove the numerically obtained dimension formula in this special case and to give explicit formulas of the locally supported basis functions. In addition, we will present a possible extension of our approach to more general volumetric two-patch domains. We conclude the thesis in Chapter 7.
Chapter 2

Characterization of $C^1$-smooth isogeometric functions

We restrict ourselves to a two-patch domain $\Omega = \Omega^{(1)} \cup \Omega^{(2)}$, as in Section 1.2 and assume that the trivariate subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ are topologically equivalent to cuboids. Moreover, they possess a common face $\Gamma$, parameterized by the two copies of the face $\hat{\Gamma} = [0,1]^2 \times \{0\}$, and are described by parameterizations $F^{(i)} : [0,1]^3 = \hat{\Omega} \to \Omega^{(i)}, i \in \{1,2\}$, see Figure 2.1.

Using the notation of Section 1.2, we find that the parametric representations $F^{(1)}$ and $F^{(2)}$ have coordinate functions, which are elements of the tensor-product spline space

$$\mathcal{P} = S^{p}_{1,k} \otimes S^{p}_{1,k} \otimes S^{p}_{1,k},$$

which consists of $C^1$–smooth trivariate spline functions. They define the trivariate

two-patch geometry mapping

$$F = (F^{(1)}, F^{(2)}) \in \mathcal{P}^3 \times \mathcal{P}^3.$$
Figure 2.1: Parameterizations of two volumetric subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ with the same parameter domain, joined at the interface $\Gamma$.

The subdomain parameterizations satisfy

$$F^{(1)}(u, v, 0) = F^{(2)}(u, v, 0) = F_0(u, v), \quad u, v \in [0, 1].$$

(2.1)

*Isogeometric discretization spaces* on the two-patch domain $\Omega$,

$$(\mathcal{P} \times \mathcal{P}) \circ F^{-1},$$

are constructed by composing pairs of spline functions belonging to $\mathcal{P} \times \mathcal{P}$ with the inverse of the geometry mapping $F$. Additional constraints are needed to construct smooth isogeometric discretizations. More precisely, we use *gluing data* to identify a suitable subspace of the full Cartesian product $\mathcal{P} \times \mathcal{P}$.

Let $\mathcal{S}_{0,k}$ denote the space of spline functions on $[0, 1]$ of unbounded degree with $k$ uniformly distributed inner knots, which are at least $C^0$ smooth. We consider four
bivariate real-valued piecewise polynomial functions

\[ \beta, \gamma, \alpha^{(1)}, \alpha^{(2)} \in S_{0,k} \otimes S_{0,k} \]

defined on \([0,1]^2\).

The gluing data

\[ D = (\beta, \gamma, \alpha^{(1)}, \alpha^{(2)}) \in (S_{0,k} \otimes S_{0,k})^4 \]

is the quadruple consisting of these four bivariate spline functions. We say that some gluing data \( D \) is regular, if both \( \alpha^{(1)} \) and \( \alpha^{(2)} \) are pointwise non-zero, i.e., these functions satisfy the inequality

\[ \alpha^{(1)}(s,t) \cdot \alpha^{(2)}(s,t) \neq 0 \quad \forall (s,t) \in [0,1]^2. \]

Given some gluing data \( D \), which may or may not be regular, we define the glued spline space

\[ G_D = \left\{ f = (f^{(1)}, f^{(2)}) \in \mathcal{P}^2 : \begin{array}{c} f^{(1)} = f^{(2)} \text{ on } \Gamma \\ \beta \partial_u f^{(1)} - \gamma \partial_v f^{(1)} + \alpha^{(2)} \partial_w f^{(1)} - \alpha^{(1)} \partial_w f^{(2)} = 0 \text{ on } \Gamma \end{array} \right\}, \]

which is the subspace of the Cartesian product \( \mathcal{P} \times \mathcal{P} \) consisting of pairs of functions that satisfy the continuity condition \((a)\), which involves the values of both functions on the interface, and the additional derivative compatibility condition \((b)\).

The partial derivative operator \( \partial_u \) indicates the differentiation with respect to the \( u \)-direction, similarly for \( \partial_v \) and \( \partial_w \) and the \( v \)- and \( w \)-direction, respectively.

We briefly discuss two special cases:
Characterization of $C^1$-smooth isogeometric functions

- When choosing trivial data $D_0 = (0, 0, 0, 0) = 0$, the compatibility condition is satisfied for any choice of the two functions, hence we obtain the space $\mathcal{G}_0$ consisting of pairs of spline functions that are joined together with $C^0$-smoothness. Any space defined by non-trivial gluing data $D$ is contained in $\mathcal{G}_0$.

- We obtain pairs of spline functions with identical first derivatives along the interface when choosing $D_1 = (0, 0, 1, 1)$. This corresponds to the classical case of spline functions that are joined together with $C^1$-smoothness.

Both the two-patch geometry mapping and the functions used for the discretizations will be chosen from the glued spline space.

In the sequel we will use the B-spline representation

$$f^{(i)} = \sum_{k=0}^{n} b_k^{(i)} N_{i_1,i_2,i_3}^{p}(u,v,w), \quad i \in \{1, 2\},$$

of the functions $f = (f^{(1)}, f^{(2)}) \in \mathcal{P} \times \mathcal{P}$, where

$$N_{i_1,i_2,i_3}^{p} = N_{i_1}^{p}N_{i_2}^{p}N_{i_3}^{p}, \quad i_1, i_2, i_3 = 0, \ldots, p + k(p - 1) \quad (2.3)$$

are the tensor-product B-splines spanning the space $\mathcal{P}$. Consequently, any function $f \in \mathcal{G}_D$ is characterized by the two equations

$$0 = \sum_{k=0}^{n} b_k^{(1)} N_{i_1,i_2,i_3}^{p}(u,v,0) - b_k^{(2)} N_{i_1,i_2,i_3}^{p}(u,v,0) \quad \text{and}$$

$$0 = \sum_{k=0}^{n} b_k^{(1)} \left( \beta(u,v) (\partial_u N_{i_1,i_2,i_3}^{p})(u,v,0) - \gamma(u,v) (\partial_v N_{i_1,i_2,i_3}^{p})(u,v,0) \right)$$

$$+ \alpha^{(2)}(u,v) (\partial_w N_{i_1,i_2,i_3}^{p})(u,v,0) \right) - b_k^{(2)} \alpha^{(1)}(u,v) (\partial_w N_{i_1,i_2,i_3}^{p})(u,v,0). \quad (2.4)$$
2.1 Geometrically continuous isogeometric functions

We consider geometry mappings \( F \in G^3_0 \subset (P \times P)^3 \), hence the \( C^0 \) condition (2.1) is satisfied. Therefore the pairs of functions satisfy \( (F^{(1)}_j, F^{(2)}_j) \in G_0, j \in \{1, 2, 3\} \). For each mapping \( F \) we are interested in the space

\[
\mathcal{V}_F = \left[ (P \times P) \circ F^{-1} \right] \cap C^1(\Omega^{(1)} \cup \Omega^{(2)})
\]

of \( C^1 \)-smooth isogeometric functions on the computational domain \( \Omega \).

As in the bivariate case, we consider isogeometric functions \( \phi \in (P \times P) \circ F^{-1} \). For each isogeometric function there exist two spline functions \( g^{(1)}, g^{(2)} \in P \) such that

\[
(\phi|_{\Omega^{(i)}})(\xi) = \phi^{(i)}(\xi) = (g^{(i)} \circ (F^{(i)})^{-1})(\xi), \quad \xi = (\xi_1, \xi_2, \xi_3) \in \Omega^{(i)}, \ i \in \{1, 2\},
\]

or equivalently

\[
(\phi|_{\Omega^{(i)}})(F^{(i)}(x)) = \phi(F^{(i)}(x)) = g^{(i)}(x), \quad x = (u, v, w) \in \hat{\Omega} = [0,1]^3, \ i \in \{1, 2\}.
\]

In order to guarantee that \( \phi \in C^1(\Omega^{(1)} \cup \Omega^{(2)}) \), hence that the first derivatives of \( \phi^{(1)} \) and \( \phi^{(2)} \) take identical values at the interface \( F_0(u, v) \), we consider the associated graph hypersurface in four-dimensional space

\[
\Sigma = (\Sigma^{(1)}, \Sigma^{(2)}) \in P^4 \times P^4;
\]

with

\[
\Sigma^{(i)} = \left( F^{(i)}, g^{(i)} \right)^T : \hat{\Omega} \to \mathbb{R}^4, \ i \in \{1, 2\}.
\]
The function $\phi$ is $C^1$–smooth if and only if the two patches have identical tangent hyperplanes along the interface

$$\Sigma^{(1)}(u,v,0) = \Sigma^{(2)}(u,v,0) = \Sigma_0(u,v), \quad u,v \in [0,1],$$

(2.5)

see Theorem 1 in [36] for the bivariate case. This is characterized by two conditions: First, the spline functions $g^{(i)}$ satisfy the $C^0$ condition

$$g^{(1)}(u,v,0) = g^{(2)}(u,v,0) = g_0(u,v), \quad u,v \in [0,1].$$

Second, the four partial derivatives

$$\partial_u \Sigma_0(u,v), \partial_v \Sigma_0(u,v), \partial_w \Sigma^{(1)}(u,v,0) \quad \text{and} \quad \partial_w \Sigma^{(2)}(u,v,0)$$

(2.6)

of both patches are linearly dependent at each point of the interface (2.5). These conditions are equivalent to the well-established notion of geometric continuity between patches (see [23]), hence we refer to $\mathcal{V}_F$ as the space of $C^1$–smooth geometrically continuous isogeometric functions.

The linear dependency can equivalently be characterized by the vanishing determinant of the $4 \times 4$ matrix consisting of the vectors in (2.6). The resulting equation

$$J(u,v) = \det \begin{pmatrix} \nabla F^{(1)}|_{w=0} & \partial_w F^{(2)}|_{w=0} \\ \nabla g^{(1)}|_{w=0} & \partial_w g^{(2)}|_{w=0} \end{pmatrix} = 0$$

(2.7)

generalizes the well–known determinant condition for geometric continuity of surfaces [26]. We consider the cofactor expansion along the last row and obtain

$$J_{41} \partial_u g_0 - J_{42} \partial_v g_0 + J_{43} (\partial_w g^{(1)}|_{w=0}) - J_{44} (\partial_w g^{(2)}|_{w=0}) = 0.$$
We note that
\[ J_{44} = \det \nabla F^{(1)}|_{w=0} \quad \text{and} \quad J_{43} = \det \nabla F^{(2)}|_{w=0}. \]

These minors define the \textit{geometric gluing data}
\[ D_F = (J_{41}, J_{42}, J_{43}, J_{44}). \] (2.8)

We call it geometric as it is obtained from a geometry mapping \( F \). Two observations are in order:

- The coordinate functions of the two-patch geometry mapping \( F \) are elements of the glued spline space defined by the geometric gluing data \( D_F \), i.e.,
\[ F \in G^{3}_{D_F}. \]

- The pair \( g = (g^{(1)}, g^{(2)}) \) of spline functions is contained in the glued spline space defined by \( D_F \) if and only if the corresponding isogeometric function \( \phi \) is \( C^1 \)–smooth, i.e., \( \phi \in V_F \). Thus any \( C^1 \)–smooth geometrically continuous isogeometric function is simply the push-forward of a glued spline function, i.e.,
\[ V_F = G_{D_F} \circ F^{-1}. \] (2.9)

So far, we considered geometric gluing data \( D_F \) defined by a given geometry mapping \( F \). Now we will start from general gluing data \( D \) and use it to define a geometry mapping. We will see, this mapping yields geometric gluing data that is essentially equivalent to \( D \) (see also [15] for the bivariate case):
Lemma 2.1. We consider regular gluing data $D = (\beta, \gamma, \alpha^{(1)}, \alpha^{(2)})$ and a regular two-patch geometry mapping $F \in G_D^3$. The geometric gluing data $D_F$ defined by $F$ satisfies

$$D_F = \zeta D$$

with the non-zero factor

$$\zeta(u, v) = \frac{J_{44}}{\alpha^{(1)}(u, v)}.$$ 

Proof. The fact that $F \in G_D^3$ implies

$$\partial_w F^{(2)}|_{w=0} = \frac{\beta}{\alpha^{(1)}} \partial_u F_0 - \frac{\gamma}{\alpha^{(1)}} \partial_v F_0 + \frac{\alpha^{(2)}}{\alpha^{(1)}} \left( \partial_w F^{(1)}|_{w=0} \right),$$

since the regularity of the gluing data ensures $\alpha^{(1)} \neq 0$.

Substituting the right-hand side of this equation into (2.7) and expanding the determinant with respect to the last row gives minors

$$J_{41} = \frac{\beta}{\alpha^{(1)}} J_{44}, \quad J_{42} = \frac{\gamma}{\alpha^{(1)}} J_{44}, \quad J_{43} = \frac{\alpha^{(2)}}{\alpha^{(1)}} J_{44}. \quad (2.11)$$

The regularity of the geometry mapping implies $J_{44} \neq 0$ at all points, hence combining (2.11) with the definition (2.8) of the geometric gluing data $D_F$ completes the proof.

As an immediate consequence of (2.10) we obtain that the two glued spline spaces are identical,

$$G_{D_F} = G_D$$

since the glued spline spaces defined by $D$ and $\zeta D$ are identical for any non-zero factor $\zeta$.

Groisser and Peters proved in [23] that $G^k$ constructions always yield $C^k$–smooth
isogeometric functions, where they considered $G^k$-smoothness of $m$-variate geometry mappings into $d$-dimensional spaces and functions with values in $\mathbb{R}^N$. When applying it to our specific case ($k = 1, m = 3, d = 3$ and $N = 1$), it allows to conclude that

$$G_D \circ F^{-1} \subseteq V_F.$$ 

We can use our previous results to re-derive the Theorem of Groisser and Peters in our particular situation ($C^1$-smooth isogeometric trivariate functions):

**Theorem 2.2.** Consider regular gluing data $D$ and a regular geometry mapping $F \in G^3_D$. Any $C^1$-smooth geometrically continuous isogeometric function is the push-forward of a glued spline function,

$$V_F = G_D \circ F^{-1}.$$ 

**Proof.** This can be seen by combining (2.9) and (2.12). \[  \]

The above Theorem 2.2 complements the observation of Groisser and Peters by establishing the equivalence of both spaces under an additional regularity assumption. We conjecture that it can be extended to the general case as well. This, however, would require more technicalities and is therefore beyond the scope of the present work.

### 2.2 Construction of non-trivial spaces $V_F$

We analyze the dimension of the space of $C^1$-smooth geometrically continuous isogeometric splines $V_F$ for any regular geometry mapping $F \in G^3_D$ — and hence, in
view of Theorem 2.2, the dimension of the glued spline space $\mathcal{G}_D$— for specific types of gluing data.

In the following we define several types of gluing data and introduce the *generic dimension* of the associated space of $C^1$–smooth isogeometric functions. In particular, from now on we restrict ourselves to polynomial (rather than piecewise polynomial) gluing data.

In general, the dimension is very low. In particular, due to the linear precision of tensor-product B-splines, the space of $C^1$– smooth geometrically continuous isogeometric functions always contains the four-dimensional space of linear polynomials, but it does not contain any additional function in general.

In the special case that the gluing data is derived from a regular trilinear geometry mapping $\mathbf{F} \in \mathcal{G}_D^3$, the space $\mathcal{V}_F$ is guaranteed to contain any polynomial $\phi$ of (total) degree $p$. In fact, we have $\phi \in \mathcal{V}_F$ since

$$\phi = (g^{(1)}, g^{(2)}) \circ \mathbf{F}^{-1}$$

for the two tensor-product polynomials (hence also spline functions)

$$(g^{(1)}, g^{(2)}) = \phi \circ \mathbf{F} = (\phi \circ \mathbf{F}^{(1)}, \phi \circ \mathbf{F}^{(2)}) \in \mathcal{P} \times \mathcal{P}$$

of degree $p$. The total degree does not exceed $p$ since $\mathbf{F}$ is trilinear. Therefore the dimension of $\mathcal{V}_F$ is at least $\binom{p+3}{3}$ in this situation.

In order to obtain a space, whose dimension even grows as the number $k$ of inner knots is increased, we consider polynomial gluing data of relatively low degree, i.e.,

$$D = (\beta, \gamma, \alpha^{(1)}, \alpha^{(2)}) \in \Pi^{q_1} \times \Pi^{q_2} \times \Pi^{q_3} \times \Pi^{q_4},$$
where \( \Pi^q \) denotes the space of bivariate tensor-product polynomials of bi-degree \( q_i \in \mathbb{Z}_{\geq 0}^2 \). The four bi-degrees \( Q = [q_1, q_2, q_3, q_4] \) thus characterize a class of gluing data.

In particular, if we consider geometric gluing data, derived from a generic trilinear geometry mapping, then due to (2.8) we obtain for \( \beta \) the bi-degree \( q_1 = \deg \beta = \deg J_{41} = (3, 2) \). This can be seen by considering the cofactor expansion along the last row

\[
\beta = \begin{vmatrix}
\partial_u F_1^{(1)}(u, v, 0) & \partial_v F_1^{(1)}(u, v, 0) & \partial_w F_1^{(2)}(u, v, 0) \\
\partial_v F_2^{(1)}(u, v, 0) & \partial_w F_2^{(1)}(u, v, 0) & \partial_w F_2^{(2)}(u, v, 0) \\
\partial_w F_3^{(1)}(u, v, 0) & \partial_w F_3^{(1)}(u, v, 0) & \partial_w F_3^{(2)}(u, v, 0)
\end{vmatrix},
\]

taking the derivatives and considering the maximal bi-degree with respect to \((u, v)\).

In the same way one obtains the remaining bi-degrees for \( \gamma, \alpha^{(1)} \) and \( \alpha^{(2)} \). Therefore this trilinear geometric gluing data has degree

\[
Q_3 = [(3, 2), (2, 3), (2, 2), (2, 2)].
\] (2.13)

Our goal is to analyze the dimension of \( \mathcal{V}_F \) for gluing data with respective degree not exceeding \( Q_3 \). Let \( \mathcal{D} \) be the space of gluing data of degree at most \( Q_3 \), which has dimension

\[
\dim \mathcal{D} = \dim \Pi^{(3,2)} + \dim \Pi^{(2,3)} + 2 \dim \Pi^{(2,2)} = 42.
\]

In addition to considering the full 42-dimensional space, we will also analyze the dimension for gluing data \( D \) taken from certain subsets of \( \mathcal{D} \). These subspaces are
defined as rational embeddings

\[ \mathcal{T}_{\text{type}} : \mathbb{R}^m \to \mathcal{D}, \]

of an \( m \)-dimensional real linear space \( \mathbb{R}^m \), for some dimension \( m \in \mathbb{Z}_{>0} \). The subscript “type” identifies a certain class of gluing data.

Each class is characterized by a set of algebraic equations over the coefficients of the gluing data and, consequently, the associated images

\[ \mathfrak{T}_{\text{type}} = \mathcal{T}_{\text{type}}(\mathbb{R}^m) \]

form algebraic varieties in \( \mathcal{D} \equiv \mathbb{R}^{42} \).

### 2.3 Types of gluing data

More precisely we consider the following embeddings:

1) The corner points \( c_{j,k,\ell} \) of the volumetric patches (which are topologically equivalent to cuboids) define a trilinear two-patch geometry mapping

\[ F = (F^{(1)}, F^{(2)}), \]

\[ F^{(i)} = \sum_{j=0}^{1} \sum_{k=0}^{1} \sum_{\ell=0}^{1} c_{j,k,(-1)^{i+1}} \beta_j(u) \beta_k(v) \beta_\ell(w), \quad (2.14) \]

where the blending functions \( \beta_j, \beta_k, \beta_\ell \) are the linear Bernstein polynomials, see Figure 2.2.

We obtain the geometric gluing data by evaluating the subdeterminants \( J_{4m} \).
Types of gluing data

Figure 2.2: The two-patch domain Ω with the two trilinearly parameterized subdomains Ω(1), Ω(2), which are joined at the common interface Γ and their defining vertices.

as in (2.7). The mapping

\[ \mathcal{T}_{\text{trl}} : \mathbb{R}^{36} \rightarrow D, \quad F \mapsto (J_{41}, J_{42}, J_{43}, J_{44}), \]

which transforms the 36 coordinates of the 12 corner points into the associated geometric gluing data, defines the algebraic variety \( \mathcal{S}_{\text{trl}} \) of \textit{trilinear} geometric gluing data.

Note that in fact we have \( \dim(\mathcal{P} \times \mathcal{P})^3 = 48 \). Since we assume \( F^{(1)}(u, v, 0) = F^{(2)}(u, v, 0) \), we identify the corner points of the shared interface of both patches to be equal and obtain the 36 coordinates of the remaining 12 corner points that define the two-patch geometry.

2) Considering again the situation of the trilinear embedding, we obtain the special case of planar interfaces\(^1\) by choosing corner points \( c_{j,k,0} \in \mathbb{R}^2 \times \{0\} \).

\(^1\)Without loss of generality we choose the plane with \( \xi_3 = 0 \) as the one containing the interface.
The mapping

\[ \mathcal{T}_{\text{pln}} : \mathbb{R}^{32} \to \mathcal{D}, \]  

(2.15)

which transforms the 32 free coordinates of the 12 corner points into the associated geometric gluing data, defines the algebraic variety \( \mathcal{T}_{\text{pln}} \subseteq \mathcal{T}_{\text{trl}} \) of trilinear geometric gluing data with a planar interface.

3) Even more restrictive choices on the 12 corner points define further types of gluing data. As an example, we focus on configurations which are symmetric with respect to the interface plane and obtain the embedding

\[ \mathcal{T}_{\text{sym}} : \mathbb{R}^{20} \to \mathcal{D}, \]  

(2.16)

defining the variety \( \mathcal{T}_{\text{sym}} \subseteq \mathcal{T}_{\text{pln}} \).

4) In the simplest possible situation, we arrive at a two-patch domain that consists of cubes with some edge length \( \delta > 0 \). This corresponds to the classical case of uniform \( C^1 \)–smooth splines. The associated geometric gluing data defines the embedding

\[ \mathcal{T}_{\text{uni}} : \mathbb{R} \to \mathcal{D}, \]  

(2.17)

which specifies the variety \( \mathcal{T}_{\text{uni}} \subseteq \mathcal{T}_{\text{sym}} \).

5) For the sake of completeness we also include the most general case of using generic gluing data \( \mathcal{D} \) of degree \( Q_3 \) as in (2.13). In the following we will denote such non-trilinear gluing data of degree \( Q_3 \) as cubic gluing data. By considering the 42 coefficients of these four tensor-product polynomials we find
the embedding

\[ T_{\text{cub}} : \mathbb{R}^{42} \to D, \quad (2.18) \]

which defines the full set \( \mathcal{X}_{\text{cub}} = D \).

6) Considering again the situation of the generic embedding, we obtain the special cases of \textit{quadratic} and \textit{linear} gluing data by restricting some of the coefficients of the four polynomials \( \beta, \gamma, \alpha^{(1)}, \alpha^{(2)} \) to zero. In particular, we consider the degrees

\[ Q_2 = [(2, 1), (1, 2), (1, 1), (1, 1)] \quad \text{and} \quad Q_1 = [(1, 0), (0, 1), (0, 0), (0, 0)], \]

respectively and obtain the embeddings

\[ T_{\text{quad}} : \mathbb{R}^{20} \to D, \quad (2.19) \]

and

\[ T_{\text{lin}} : \mathbb{R}^{6} \to D \quad (2.20) \]

which define the varieties \( \mathcal{X}_{\text{lin}} \subseteq \mathcal{X}_{\text{quad}} \subseteq \mathcal{X}_{\text{cub}} \).

Figure 2.3 summarizes the relation between the algebraic varieties in the space \( D \) which correspond to the various types of gluing data described above.
Figure 2.3: The relations between the algebraic varieties $\mathfrak{T}_{\text{type}}$, which are defined by different types of gluing data $D$. 
Chapter 3

Dimension of the space $\mathcal{V}_F$

In the following we analyze the dimension of $\mathcal{V}_F$ for an entire class of gluing data — which is defined by one of the types introduced in the previous section.

3.1 The generic dimension

For given gluing data $D \in Q_3$ we compute the dimension of the space of $C^1$-smooth isogeometric splines $\mathcal{V}_F$ by analyzing the equations (2.4). The right-hand sides of both equations are contained in the spline space $\mathcal{P}' = S_{0,k}^{p+2} \otimes S_{0,k}^{p+2}$. Thus we obtain an equivalent homogeneous linear system\(^2\)

$$A_D b = 0$$  \hspace{1cm} (3.1)

for the unknowns

$$b = \left( b_j^{(i)} \right)_{j=1,...,n, \ i \in \{1,2\}}$$  \hspace{1cm} (3.2)

\(^2\)In practice, it suffices to analyze the smaller system obtained by identifying the degrees of freedom along the common interface and restricting the system to the three layers of coefficients around it.
Dimension of the space $\mathcal{V}_F$

by performing collocation at the Greville points of $\mathcal{P}'$.

The dimension of the space of $C^1$–smooth geometrically continuous isogeometric functions is determined by the rank of the coefficient matrix,

$$\dim \mathcal{V}_F = 2n - \text{rank } A_D.$$

For fixed values of $p$, the spline degree, and $k$, the number of inner knots, the dimension defines a function on the 42-dimensional space $\mathcal{D}$.

In order to compute the dimension, we evaluate the rank of the matrix $A_D$. We pay special attention to avoid numerical errors during the computation. To do so, we use rational numbers throughout the process. The Greville abscissas have rational coordinates and the values of B-spline functions on them are also rational, which is why the matrix $A_D$ is a matrix with rational elements as well. Later, in order to obtain a basis, we will also compute the kernel of the matrix $A_D$ and analyze the structure of the solutions. More precisely we apply Gaussian elimination to transform $A_D$ to Reduced Row Echelon Form (RREF).

Recall that the rank of the coefficient matrix satisfies

$$\text{rank } A_D = \min_{s \in \mathbb{N}} \left\{ s : \text{all } (s + 1) \times (s + 1) \text{ minors of } A_D \text{ vanish} \right\}.$$

Consequently, when adopting the notion of degeneracy loci from [20],

$$\mathcal{L}_s = \{ D \in \mathcal{D} : \text{rank } A_D \leq s \},$$

it is obvious that these loci form a nested sequence of algebraic varieties,

$$\{(0, 0, 0, 0)\} = \mathcal{L}_0 \subseteq \cdots \subseteq \mathcal{L}_s \subseteq \mathcal{L}_{s+1} \subseteq \cdots \subseteq \mathcal{L}_{2n} = \mathcal{D}.$$
The generic dimension

The dimension of $\mathcal{V}_F$ is at least $2n - s$ if the degeneracy locus $\mathcal{L}_s$ contains the gluing data $D \in \mathcal{D}$.

We define the generic dimension $\delta_{\text{type}}$ as

$$\delta_{\text{type}} = 2n - \min_s \{ s : \mathcal{T}_{\text{type}} \subseteq \mathcal{L}_s \}.$$ 

This notion is justified by the following observation:

**Theorem 3.1.** For randomly chosen gluing data $D \in \mathcal{T}_{\text{type}}$, where type is one of the types introduced in Section 2.2, the dimension of the space of $C^1$-smooth geometrically continuous isogeometric functions satisfies

$$\dim \mathcal{V}_F = \delta_{\text{type}}$$

with probability 1.

**Proof.** Recall that the degeneracy loci can be characterized by the vanishing minors of the coefficient matrix $A_D$. More precisely, the degeneracy loci satisfy

$$\mathcal{L}_s = \{ D : \ell_s(D) = 0 \},$$

where $\ell_s(D)$ is the vector whose elements are all the $(s + 1) \times (s + 1)$-minors of $A_D$. Consequently, the inclusion $\mathcal{T}_{\text{type}} \subseteq \mathcal{L}_s$ can equivalently be characterized by

$$\ell_s \circ \mathcal{T}_{\text{type}} = 0 \text{ on } \mathbb{R}^m,$$

where $m$ is the dimension of the preimage space of the embeddings $\mathcal{T}_{\text{type}}$.

We consider the randomly chosen gluing data $D \in \mathcal{T}_{\text{type}}$. For any $s$, the equations $\ell_s \circ \mathcal{T}_{\text{type}} = 0$ define an algebraic variety, which is either the entire space $\mathbb{R}^m$, or
it is a lower-dimensional subvariety thereof. The latter case occurs if and only if \( s < 2n - \delta_{\text{type}} \), due to the definition of the generic dimension.

Consequently, for any randomly generated point \( x \in \mathbb{R}^m \),

\[
P\left( (\ell_s \circ T_{\text{type}})(x) = 0 \right) = \begin{cases} 
1, & \text{if } T_{\text{type}} \subseteq \mathcal{L}_s \\
0, & \text{otherwise,}
\end{cases}
\]

since lower dimensional subvarieties of \( \mathbb{R}^m \) have zero volume. Thus

\[
P\left( \min_s \{(\ell_s \circ T_{\text{type}})(x) = 0\} = \min_s (T_{\text{type}} \subseteq \mathcal{L}_s) \right) = 1
\]

which implies

\[
P(\dim V_F = \delta_{\text{type}}) = 1
\]

for randomly chosen gluing data \( D \in T_{\text{type}} \), since the embedding \( T_{\text{type}} \) transforms any point \( x \) into gluing data of the corresponding type and this completes the proof. \( \square \)

### 3.2 Dimensions for different types of gluing data

Following the approach in [36, Theorem 4], we construct a basis of the space \( V_F \) – and consequently of the space \( G_D \) – by splitting the space into a direct sum of two subspaces, i.e.

\[
V_F = V_F^\Gamma \oplus V_F^S. \tag{3.3}
\]

The first subspace \( V_F^\Gamma \) denotes the \textit{space of interface functions}. It contains the functions with non-zero coefficients on the shared face \( \Gamma \) and the two neighboring layers, see red points in Figure 3.1. These functions are affected by the specific choice of
Dimensions for different types of gluing data concerned. The second subspace $V_S^F$, referred to as the *space of standard functions*, contains functions with zero coefficients on these three layers, hence they have their support solely on the coefficients marked in blue in Figure 3.1.

The space of standard functions is spanned by the “usual” isogeometric basis functions and therefore does not depend on the choice of gluing data, since their values and first derivatives vanish along the interface. The dimension of the space of standard functions is

$$\dim V_S^F = \delta^S = 2(p + 1 + k(p - 1))^{2(p - 1 + k(p - 1))}. \quad (3.4)$$

We analyze the dimension of the interface functions as a vector space by studying randomly generated instances. Each instance provides a lower bound on the dimen-
Dimension of the space $\mathcal{V}_F$

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Table 3.1: Dimensions $\delta^S + \delta_{\text{cub}}^\Gamma$ of the space of $C^1$-smooth isogeometric splines for cubic gluing data.

sion. However, since we use random instances, the sum of both dimensions

$$\delta_{\text{type}} = \delta^S + \delta_{\text{type}}^\Gamma$$

is equal to the generic dimension with probability 1 due to the Theorem 3.1.

We now proceed by discussing $\delta_{\text{type}}^\Gamma$ for the various types of gluing data individually.

Type ‘cub’ First we consider general cubic gluing data, see (2.18). Table 3.1 presents the dimensions $\delta_{\text{cub}}$ for different degrees $(p,p,p)$ and numbers $k$ of inner knots. The dimension of the space $\mathcal{V}_F$ is shown as the sum of the dimensions of the spaces of standard and interface functions. While the dimension of the standard functions grows with $k$ for any degree $p$, the number of linearly independent interface functions remains constant for $p = 2, 3$.

We use interpolation to obtain a closed formula for the dimension of the interface functions. One expects to obtain a closed polynomial expression for sufficiently large degrees $p$, but not necessarily for all degrees. In the case of generic cubic gluing data, we did not obtain a closed formula when considering degrees $p \leq 4$. For $p > 4$, we
Table 3.2: Dimensions $\delta^S + \delta^\Gamma_{trl}$ of the space of $C^1$–smooth isogeometric splines for trilinear geometric gluing data.

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<td>2</td>
<td>150+10</td>
<td>768+40</td>
<td>2178+106</td>
<td>4704+208</td>
<td>8670+346</td>
</tr>
<tr>
<td>3</td>
<td>288+10</td>
<td>1600+53</td>
<td>4704+157</td>
<td>10368+325</td>
<td>19360+557</td>
</tr>
<tr>
<td>4</td>
<td>490+10</td>
<td>2880+68</td>
<td>8670+218</td>
<td>19360+468</td>
<td>36450+818</td>
</tr>
</tbody>
</table>

conjecture that the dimension satisfies

$$\delta^\Gamma_{cub} = -6 - 14k + 5k^2 - 10k(1 + k)p + 2(k + 1)^2p^2.$$  \hfill (3.5)

**Type ‘trl’** Table 3.2 presents the dimensions $\delta_{trl}$ for the case of trilinear geometric gluing data, see (2.14). The dimensions are always larger than in the cubic case. This is significantly different from the results in [15], for the bivariate case, where it was shown that the dimension does not change when generalizing bilinear parameterizations to analysis-suitable $G^1$ parameterizations, see Remark 3.1 below for more details.

Again we use interpolation to obtain a closed formula for the dimension of the interface functions. For $p > 2$, we conjecture that the dimension satisfies

$$\delta^\Gamma_{trl} = 2 + 2k + 13k^2 - 10k(1 + k)p + 2(k + 1)^2p^2.$$ \hfill (3.5)

**Type ‘pln’** This class of gluing data is derived from trilinear parameterizations with planar interfaces, see (2.15). We obtain exactly the same results as in the previous case (Table 3.2), hence $\delta_{pln} = \delta_{trl}$. 
Table 3.3: Dimensions $\delta^S + \delta^\Gamma_{\text{sym}}$ of the space of $C^1$-smooth isogeometric splines for trilinear gluing data corresponding to two symmetric trilinear subdomains.

**Type ‘sym’** An even more restricted class of gluing data is derived from symmetric trilinear parameterizations. Table 3.3 lists the dimensions $\delta_{\text{sym}}$ for the embedding in (2.16) as the sum of $\delta^S + \delta^\Gamma_{\text{sym}}$. The dimensions are always larger than in the previous three cases.

Again we use interpolation to obtain a closed formula for $\delta^\Gamma_{\text{sym}}$. For $p > 2$, we conjecture that the dimension satisfies

$$\delta^\Gamma_{\text{sym}} = 3 + 10k^2 + 2 (1 - 3k - 4k^2) p + 2 (k + 1)^2 p^2.$$

**Type ‘uni’** Table 3.4 presents the dimensions $\delta_{\text{uni}}$ for the case of gluing data that corresponds to two cubes, each with some edge length $\delta > 0$, with a shared face, see (2.17). The obtained dimensions are always larger than for the other cases.

We arrive at the closed formula for the dimension of the interface functions,

$$\delta^\Gamma_{\text{uni}} = 2 (1 - k)^2 + 4 (1 - k^2) p + 2 (k + 1)^2 p^2,$$

which is equal to the dimension of the interface functions among the trivariate tensor-product splines on the two cubes.
Dimensions for different types of gluing data

Table 3.4: Dimensions $\delta^S + \delta^\Gamma_{\text{uni}}$ of the space of $C^1$–smooth isogeometric splines for gluing data $\mathcal{T}_{\text{uni}}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p=2$</th>
<th>$p=3$</th>
<th>$p=4$</th>
<th>$p=5$</th>
<th>$p=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18+18</td>
<td>64+32</td>
<td>150+50</td>
<td>288+72</td>
<td>490+98</td>
</tr>
<tr>
<td>1</td>
<td>64+32</td>
<td>288+72</td>
<td>768+128</td>
<td>1600+200</td>
<td>2880+288</td>
</tr>
<tr>
<td>2</td>
<td>150+50</td>
<td>768+128</td>
<td>2178+242</td>
<td>4704+392</td>
<td>8670+578</td>
</tr>
<tr>
<td>3</td>
<td>288+72</td>
<td>1600+200</td>
<td>4704+392</td>
<td>10368+648</td>
<td>19360+968</td>
</tr>
<tr>
<td>4</td>
<td>490+98</td>
<td>2880+288</td>
<td>8670+578</td>
<td>19360+968</td>
<td>36450+1458</td>
</tr>
</tbody>
</table>

Table 3.5: Dimensions $\delta^S + \delta^\Gamma_{\text{qud}}$ of the space of $C^1$–smooth isogeometric splines for gluing data of degree $Q_2$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$p=2$</th>
<th>$p=3$</th>
<th>$p=4$</th>
<th>$p=5$</th>
<th>$p=6$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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<td>64+23</td>
<td>150+39</td>
<td>288+59</td>
<td>490+83</td>
<td>768+111</td>
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<tr>
<td>1</td>
<td>64+11</td>
<td>288+34</td>
<td>768+77</td>
<td>1600+137</td>
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<td>2</td>
<td>150+11</td>
<td>768+47</td>
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<td>3</td>
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<td>1600+62</td>
<td>4704+189</td>
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<td>19360+653</td>
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<td>490+11</td>
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<td>19360+563</td>
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<td>61440+1463</td>
</tr>
</tbody>
</table>

Types ‘qud’ and ‘lin’ Table 3.5 presents the dimensions $\delta_{\text{qud}}$ for gluing data $D$ of degree $Q_2$, see (2.19). The obtained dimensions are always larger than in case of cubic gluing data. Again we use interpolation to obtain a closed formula for the dimension of the interface functions. For $p > 3$, we conjecture that the dimension satisfies

$$
\delta^\Gamma_{\text{qud}} = -1 - 8k + 6k^2 + 2(1 - 3k - 4k^2)p + 2(k + 1)^2p^2.
$$

Table 3.6 presents the dimensions $\delta_{\text{lin}}$ for gluing data $D$ of degree $Q_1$, see (2.20).
Table 3.6: Dimensions $\delta^S + \delta^\Gamma_{\text{lin}}$ of the space of $C^1$–smooth isogeometric splines for gluing data of degree $Q_1$.

The obtained dimensions are always larger than in case of quadratic gluing data. For $k = 0$ and any choice of $p$ we obtain the same dimension $\delta_{\text{lin}}$ as for gluing data of type ‘uni’.

For $p > 2$, we conjecture that the dimension satisfies

$$\delta^\Gamma_{\text{lin}} = 2 - 6k + 5k^2 + 2(2 - k - 3k^2)p + 2(k + 1)^2 p^2.$$  

Remark 3.1. For the trivariate isogeometric functions considered in this paper, the spaces defined by cubic gluing data (more precisely, by gluing data of degree $Q_3$, see (2.13)) and by trilinear geometric gluing data have different dimensions. This is due to the fact that the algebraic varieties defined by the embeddings $T_{\text{cub}}$ and $T_{\text{trl}}$ are different. Indeed, the second one is a proper subvariety of the first one. This is a fundamental difference to the bivariate case, where we have to consider only three blending functions, which depend solely on one variable. The two classes of gluing data in the bivariate case, which correspond to the types ‘$\text{cub}$’ and ‘$\text{trl}$’ in the trivariate one, are data of degree $(2, 1, 1)$ [15] and geometric data generated by bilinear geometry mappings [36], respectively. A short computation confirms that the two associated subvarieties are identical.
Indeed, one can prove that the subset associated with the bilinear gluing data is dense in the algebraic variety generated by the data of degree \((2, 1, 1)\). Consequently, the generic dimensions of the corresponding glued spline spaces are identical. In particular, this also applies to the particular class of analysis-suitable gluing data, which is situated between the two classes, see [30].
Chapter 4

Basis functions

In order to obtain spaces of $C^1$–smooth geometrically continuous isogeometric functions (and consequently glued spline spaces) on two-patch domains that are useful for applications, we investigate two additional questions:

1. Can we find a system of locally supported\footnote{More precisely, we wish to have basis functions with a bounded number of non-zero-coefficients, where the bound is independent of the number of inner knots $k$.} basis functions spanning the space?

2. Do these functions have good approximation properties?

We will answer the first question in this chapter, the approximation power of special types of gluing data will be the topic of Chapter 5.

4.1 Existence of a locally supported basis

In order to simplify the presentation and to eliminate the influence of boundary effects, we restrict ourselves to spaces with homogeneous $C^1$ boundary conditions.
Figure 4.1: Numbering of the three layers of coefficients of interface basis functions for \( p = 3 \) and \( k = 3 \). From left to right: coefficients left of, on and right of the interface. Green points refer to coefficients that are discarded due to boundary conditions.

This does not limit the applicability of the results to isogeometric analysis since one may always transform inhomogeneous problems into homogeneous ones. The space which is obtained by enforcing these conditions will be denoted by \( \mathcal{V}_{F,0} \). Consequently, we may discard two layers of boundary coefficients from the equations (2.4) and the equivalent linear system (3.1). We obtain the decomposition

\[
\mathcal{V}_{F,0} = \mathcal{V}_{F,0}^F \oplus \mathcal{V}_{F,0}^S.
\]

Recall, that the space \( \mathcal{V}_{F}^S \) is spanned by the standard isogeometric functions, hence so is the space \( \mathcal{V}_{F,0}^S \). We focus on the computation of interface basis functions and their structure for different types of generic gluing data and various degrees. In order to find locally supported basis functions, we order the coefficients in \( b \) (see (3.2)) according to their location. Figure 4.1 shows this ordering for a particular case.

After reordering the columns of \( A_D \) according to this numbering (and discarding boundary coefficients, see green points in Figure 4.1), we compute the coefficients of the basis functions using the RREF of the matrix. In most cases we were able to identify patterns for the kernel of the matrix, which correspond to locally supported interface basis functions.
Existence of a locally supported basis

<table>
<thead>
<tr>
<th>$p$</th>
<th>trilinear gluing data</th>
<th>cubic gluing data</th>
<th>quadratic gluing data</th>
<th>linear gluing data</th>
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<tr>
<td>2</td>
<td>×</td>
<td>×</td>
<td>×</td>
<td>(×)</td>
</tr>
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<td>3</td>
<td>✓</td>
<td>×</td>
<td>(×)</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>✓</td>
<td>(×)</td>
<td>(×)</td>
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<tr>
<td>5</td>
<td>✓</td>
<td>(×)</td>
<td>(✓)</td>
<td>✓</td>
</tr>
</tbody>
</table>

Table 4.1: This table shows the different types of bases occurring for the specific types of gluing data. × indicates that $\dim \mathcal{V}_{F,0} = 0$ and (×) that only basis functions with zero support on the shared face were obtained, hence leading to locking. Entries with ✓ refer to local bases with desired properties at the interface. (✓) signals that a basis was constructed not fulfilling all requirements for local support.

To point out the importance of these local patterns, note that the computation of the sparsest kernel vectors (that is, the functions with the smallest possible support) is NP-hard. Therefore, even for small numbers of inner knots $k$, the computations can be rather inefficient. We will make use of the coefficient patterns when considering the approximation power.

More precisely, we obtained four classes of bases, visualized by different symbols in Table 4.1:

× No locally supported interface basis functions were found, $\dim \mathcal{V}_{F,0} = 0$. In these cases, the dimension of the full space of $C^1$–smooth isogeometric spline functions does not depend on $k$.

(×) Locally supported basis functions were found but they take only zero values on the interface. This implies locking and hence a loss of approximation power.

✓ We found a local basis of the space $\mathcal{V}_{F,0}^p$, and hence of the complete space $\mathcal{V}_{F,0}$ and there are basis functions with non-zero coefficients on the interface.
The basis of the space \( \mathcal{V}_{F,0} \) has a \( k \)-dependent dimension, but there exist basis functions that are not completely local.

Only global basis functions were found for almost all types of gluing data if \( p = 2 \), and for cubic gluing data if \( p = 3 \). This matches the corresponding dimensions results of Section 3.2, which showed that the dimension of the interface functions does not increase with the number \( k \) of inner knots. Among the other cases, the most promising ones are trilinear and linear gluing data of degree \( p \geq 3 \). The case of quintic splines for quadratic gluing data is still open, since applying a re-ordering of the coefficients might still lead to a locally supported basis\(^4\).

### 4.2 Basis functions for trilinear geometric gluing data

In this section we discuss the obtained interface functions in case of trilinear geometric gluing data \( \mathcal{T}_{\text{trl}} \) and \( C^1 \) boundary conditions in further detail, focusing on degrees \( p = 3, 4 \). Similar results can be obtained for higher degrees. Independently of the spline degree \( p > 2 \), we obtain a local basis containing functions taking non-zero values on the interface, provided that there are sufficiently many inner knots. In this case the dimension of the space \( \mathcal{V}_{F,0} \) is equal to

\[
\dim \mathcal{V}_{F,0} = 10 + k (2 - 11 k) - p (2 + 2 k - 4 k^2).
\]

For applications it is important to understand the structure of the basis functions, i.e., their coefficient patterns. For degree \( p = 3 \), the basis can be generated by using

\(^4\)Using the current approach leads to basis functions which are only local in one direction.
functions of only one type, see Figure 4.2. The basis is formed by \((k - 2)^3\) functions of this type, provided that we have \(k \geq 3\), with coefficient patterns obtained by performing index shifts in \((2\mathbb{Z})^2\) (horizontally and/or vertically).

The values of these functions are visualized by an isosurface (top left) and by the level curves in a plane that intersect the interface transversally (right). These level curves are connected smoothly across the interface, thereby confirming the \(C^1\)-smoothness of the basis. Furthermore, the coefficient pattern, which involves \(6 \times 6 \times 3\) coefficients, is shown (bottom left). Note that a present ring indicates a non-zero coefficient value in the corresponding layer, where the middle ring refers to the shared interface and
Figure 4.3: Coefficient patterns of basis functions for trilinear geometric gluing data, $p = 4$. 
the inner and outer rings to the two adjacent layers of coefficients. Additionally we provide reflection lines of a particular level set (bottom right).

This approach can be generalized to basis functions of degree $p = 4$. In this case we distinguish between six types of basis functions, depending on the number of non-zero coefficients. The coefficients form seven different patterns, one for each of the Types 1 and 3-6, and two for Type 2, see Figure 4.3. Figure 4.4 visualizes these functions. The global $C^1$-smoothness is confirmed by the smoothness of the level curves (right column).

Different instances of the basis functions are obtained when using coefficient patterns obtained by performing index shifts by multiples of $p - 1 = 3$ (horizontally and/or vertically). Not all the functions obtained in this way are present in the basis of the space of $C^1$-smooth splines:

- $2k - 1$ functions of Type 4.1, with shifts in $2 \cdot 3\mathbb{Z}$,
- $k - 1$ functions of Type 4.2.1, with shifts in $3\mathbb{Z}$,
- $k(k - 1)$ functions of Type 4.2.2, with shifts in $(3\mathbb{Z})^2$, and
- $(k - 1)^2$ functions of Types 4.3–4.6, with shifts in $(3\mathbb{Z})^2$.

Summing up, the size of the basis amounts to $5k^2 - 6k + 2$, assuming that $k$ is non-zero. Note that in case of $k = 1$ a basis is spanned by functions of Type 4.1 only.

The same procedure can be applied to obtain locally supported basis functions for degrees $p > 4$. 
Figure 4.4: Isosurfaces (left column) and level curves in a plane that intersects the interface transversally (right column) visualizing the values of the six different types of basis functions for trilinear geometric gluing data, $p = 4$. 

(a) Type 1
(b) Type 2
(c) Type 3
(d) Type 4
(e) Type 5
(f) Type 6
4.3 Basis functions for linear gluing data

Recall that linear gluing data consists of four polynomials $\beta, \gamma, \alpha^{(1)}, \alpha^{(2)}$ with bi-degrees $[(1, 0), (0, 1), (0, 0), (0, 0)]$ with respect to $(u, v)$. The space of interface func-
tions has the following dimension for $p \geq 3$

$$\dim Y_{F,0}^p = \frac{1}{2} \left( 18 + 87k + 5k^2 - p(12 - 56k - 11k^2) + p^2(2 + 9k + 4k^2) \right).$$

We again construct coefficient patterns of locally supported basis functions satisfying first order homogeneous boundary conditions for $p = 3$ and $p = 4$. The results are reported in Figure 4.5. The six plots visualize the non-zero coefficients for the different types of functions. More precisely, the outer and inner rings indicate non-zero coefficients in the left and right adjacent layer of the interface, while the middle one represents a non-zero coefficient on the interface. Note that functions of Type $I$ are zero on the shared interface. The full system of basis functions is obtained by performing suitable index shifts as described below.

For $p = 3$, the basis is formed by $(2k)^2$ functions of Type $I$, using index shifts in $\mathbb{Z}^2$, and by $(k - 2)^2$ functions of Type 3, using index shifts in $(2\mathbb{Z})^2$. For $p = 4$ the basis contains the five types of functions:

- $(3k + 1)^2$ functions of Type $I$, with index shifts in $\mathbb{Z}^2$,
- $k^2$ functions of Type 4.1, with index shifts in $(3\mathbb{Z})^2$,
- $(k - 1)k$ functions of both Type 4.2 and Type 4.3, with index shifts in $(3\mathbb{Z})^2$,
- $(k - 1)^2$ functions of type 4.4, with index shifts in $(3\mathbb{Z})^2$.

As for trilinear geometric gluing data, similar results can be obtained for higher spline degrees $p$. 
Chapter 5

Approximation power of trilinear geometric gluing data

In this chapter, we exploit the fact that the local subproblems, which are defined by the shifted coefficient patterns, are known to have a kernel dimension equal to one. Consequently, it is no longer necessary to use rational arithmetic. Instead, since we know that we are looking for a single kernel vector, we use floating point computations and perform singular value decompositions (SVD). We then keep the vector associated to the singular value closest to zero.

The savings in time and memory needed for the basis computation when using floating point operations instead of exact arithmetic is demonstrated in Table 5.1 and Table 5.2. The entries of the tables refer to the construction of the basis of the space $V_{p,0}^{F}$ for spline degree $p = 3$ and different numbers of inner knots $k$. We compare the approach used in Chapter 4, where we had to set up the complete matrix obtained from (2.4) using rational arithmetic, with the more efficient construction using local subproblems and SVD based on floating point operations.
Approximation power of trilinear geometric gluing data

<table>
<thead>
<tr>
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<th>$k = 15$</th>
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<td>25.82s</td>
</tr>
<tr>
<td>Rational</td>
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<td>1037.56s</td>
<td>&gt;24h</td>
<td>&gt;72h</td>
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</tbody>
</table>

Table 5.1: Comparison of time needed, when using floating point and rational computations to construct the basis.

<table>
<thead>
<tr>
<th></th>
<th>$k = 3$</th>
<th>$k = 7$</th>
<th>$k = 15$</th>
<th>$k = 31$</th>
</tr>
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<td>Floating point</td>
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<td>11.4MBytes</td>
<td>17.6MBytes</td>
<td>42.4MBytes</td>
</tr>
<tr>
<td>Rational</td>
<td>20.8MBytes</td>
<td>426.8MBytes</td>
<td>&gt;17.7GBytes</td>
<td>&gt;32GBytes$^5$</td>
</tr>
</tbody>
</table>

Table 5.2: Comparison of memory needed, when using floating point and rational computations to construct the basis.

The entries in Table 5.1 show the time spent on the computations, whereas the values in Table 5.2 depict the maximal resident set size (RSS), which is the amount of memory occupied by the computation that is held in main memory (RAM). The expected massive advantage of the localized computation using floating point operations is clearly visible.

We explore the approximation power of the basis obtained for trilinear geometric gluing data in case of spline degree $p = 3, 4$. To determine the rates of convergence we use $L^2$-fitting of suitable target functions that are defined on the two-patch geometry $\Omega = \Omega^{(1)} \cup \Omega^{(2)}$ shown in Figure 5.1. The defining corner points of the interface are shown in red, whereas the remaining eight corner points are depicted in blue and turquois. Besides the usual norms – $H^1(\Omega), L^2(\Omega), L^\infty(\Omega)$, which are defined on the entire domain – we analyze the residuals via the following norms on the interface:

\[ \text{Aborted because of too high memory requirements.} \]
For bivariate $C^1$–smooth spline spaces, optimal convergence rates were obtained for degree $p = 3$ and higher, see [36]. However, this does not extend to the trivariate case of trilinearly parameterized two-patch domains, as shown in Figure 5.2. In the left picture, the global error is shown, where a small reduction in the approximation power can be recognized. This loss in the convergence rate is solely introduced by the error on the interface, which is depicted in the right plot.

The reduction can be explained by taking a closer look on the involved spline functions. Consider two spline functions $f, \tilde{f} \in \mathcal{G}_D$, with $f|_{\Gamma} = \tilde{f}|_{\Gamma}$. Since both functions are elements of the glued spline space $\mathcal{G}_D$, they satisfy the following equation

$$\beta \partial_u g^{(1)} - \gamma \partial_u g^{(2)} + \alpha^{(2)} \partial_u g^{(1)} - \alpha^{(1)} \partial_u g^{(2)} = 0,$$
Approximation power of trilinear geometric gluing data

Figure 5.2: $L^2$-approximation errors for trilinear geometric gluing data of degree $p = 3$.

with $g \in \mathcal{G}_D, g^{(1)} = g|_{\Omega^{(1)}}, g^{(2)} = g|_{\Omega^{(2)}},$ see (2.4).

Therefore, their difference, which is again a function in the glued spline space, satisfies

$$\beta \frac{\partial_u (f^{(1)} - \tilde{f}^{(1)})}{|\Gamma|} - \gamma \frac{\partial_u (f^{(1)} - \tilde{f}^{(1)})}{|\Gamma|} +$$

$$\alpha^{(2)} \frac{\partial_w (f^{(1)} - \tilde{f}^{(1)})}{|\Gamma|} - \alpha^{(1)} \frac{\partial_w (f^{(2)} - \tilde{f}^{(2)})}{|\Gamma|} = 0,$$

hence

$$\alpha^{(1)} \frac{\partial_w (f^{(2)} - \tilde{f}^{(2)})}{|\Gamma|} = \alpha^{(2)} \frac{\partial_w (f^{(1)} - \tilde{f}^{(1)})}{|\Gamma|}.$$

It can be observed that $\gcd(\alpha^{(1)}, \alpha^{(2)}) = 1$, see (6.1). This implies that $\alpha^{(2)}$, which is
a bivariate polynomial of degree (2,2), is a factor of each of the polynomial segments of
\[ d^{(2)} = \partial_w (f^{(2)} - \tilde{f}^{(2)})|_{\Gamma}. \]

Similarly, \( a^{(1)} \) is a factor of each of the polynomial segments of
\[ d^{(1)} = \partial_w (f^{(1)} - \tilde{f}^{(1)})|_{\Gamma}. \]

We obtain two \( C^1 \)-smooth piecewise polynomial functions \( d^{(1)}/a^{(1)} \) and \( d^{(2)}/a^{(2)} \). Since the degree of these functions does not exceed \((3,3) - (2,2) = (1,1)\), we conclude that the spline functions
\[ d^{(s)} = \partial_w (f^{(s)} - \tilde{f}^{(s)})|_{\Gamma} \]
are indeed single polynomials of degree \((3,3)\) and therefore \( C^\infty \) smooth.

Consequently, the cross-boundary derivatives of any two functions \( f \) and \( \tilde{f} \), which take the same values on the interface \( \Gamma \), differ only by a bi-cubic polynomial with only four degrees of freedom, and this does not change as \( h \) is decreased. This observation explains the loss of approximation power, as follows.

We consider two smooth functions \( \varphi, \tilde{\varphi} \in C^\infty(\Omega) \). There exist two sequences \((f_h)_h\) and \((\tilde{f}_h)_h\), whose elements are taken from the glued spline spaces obtained for element size \( h = \frac{1}{k+1} \to 0 \), such that \((f_h \circ F^{-1})_h\) and \((\tilde{f}_h \circ F^{-1})_h\) converge to \( \varphi \) and \( \tilde{\varphi} \), respectively. If these sequences converged with the full approximation power, the derivatives would converge as well, hence
\[ d^{(i)}_h = \partial_w (f^{(i)}_h - \tilde{f}^{(i)}_h)|_{\Gamma} \to \left( \partial_w \left((\varphi - \tilde{\varphi}) \circ F^{(i)}\right)\right)|_{\Gamma}, \quad i \in \{1, 2\}, \]
as \( h \to 0 \). However, this is impossible for almost all pairs of given functions \( \varphi, \tilde{\varphi} \), since the functions \( d^{(i)}_h \) are single bicubic polynomials for any \( h \).
Approximation power of trilinear geometric gluing data

![Graphs showing L^2-approximation errors for trilinear geometric gluing data of degree p = 4.](image)

<table>
<thead>
<tr>
<th>h</th>
<th>H^1(Ω)</th>
<th>L^2(Ω)</th>
<th>L^∞(Ω)</th>
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<th>L^2(Γ)</th>
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<td>0.11</td>
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<td>1.83 \cdot 10^{-2}</td>
<td>5.40 \cdot 10^{-3}</td>
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<td>1.90 \cdot 10^{-3}</td>
</tr>
<tr>
<td>8.33 \cdot 10^{-2}</td>
<td>4.83 \cdot 10^{-3}</td>
<td>3.92 \cdot 10^{-4}</td>
<td>9.49 \cdot 10^{-4}</td>
<td>3.99 \cdot 10^{-4}</td>
<td>4.66 \cdot 10^{-6}</td>
<td>6.75 \cdot 10^{-5}</td>
</tr>
<tr>
<td>4.17 \cdot 10^{-2}</td>
<td>2.89 \cdot 10^{-4}</td>
<td>1.12 \cdot 10^{-5}</td>
<td>3.76 \cdot 10^{-5}</td>
<td>2.36 \cdot 10^{-5}</td>
<td>1.17 \cdot 10^{-7}</td>
<td>2.31 \cdot 10^{-6}</td>
</tr>
</tbody>
</table>

Figure 5.3: L^2-approximation errors for trilinear geometric gluing data of degree p = 4.

If we consider spline degrees p > 3, this argument no longer applies and we observe full approximation power, as shown in Figure 5.3 for p = 4.
Chapter 6

Trilinear two-patch domains

We concentrate on the specific case of trilinear geometric gluing data $G_{D_F}$. In Chapters 3 and 4 the space of $C^1$–smooth isogeometric splines for this class of gluing data was experimentally analyzed. In the following we develop the theoretical framework to explore the space $V_F$. We will for example use the framework to prove the numerically obtained dimension (3.5) and to describe a simple explicit basis construction.

More precisely, we consider a two-patch geometry mapping $F = (F^{(1)}, F^{(2)})$, where both subdomains are described by trilinear, regular parameterizations $F^{(i)} : [0,1]^3 \rightarrow \Omega^{(i)}$, $i \in \{1,2\}$.

The two-patch domain $\Omega$ is fully determined by 12 vertices $c_{j,k,\ell}$, which are given by

$c_{0,0,-1} = F^{(1)}(0,0,1), c_{1,0,-1} = F^{(1)}(1,0,1), c_{0,1,-1} = F^{(1)}(0,1,1), c_{1,1,-1} = F^{(1)}(1,1,1),$

$c_{0,0,0} = F^{(1)}(0,0,0), c_{1,0,0} = F^{(1)}(1,0,0), c_{0,1,0} = F^{(1)}(0,1,0), c_{1,1,0} = F^{(1)}(1,1,0),$
\[ c_{0,0,1} = F^{(2)}(0,0,1), \quad c_{1,0,1} = F^{(2)}(1,0,1), \quad c_{0,1,1} = F^{(2)}(0,1,1), \quad c_{1,1,1} = F^{(2)}(1,1,1), \]

with \( i \in \{1,2\} \), see Figure 2.2. This ensures that the common face \( \Gamma \) is parameterized by

\[
F^{(1)}(u,v,0) = F^{(2)}(u,v,0) = F_0(u,v), \quad (u,v) \in [0,1]^2,
\]

see (2.1). Consequently, we consider a two-patch geometry mapping \( F \in G_{DF}^3 \).

Recall that the geometry mappings \( F^{(i)} \), \( i \in \{1,2\} \), are trilinearly parameterized, which implies that they also possess a spline representation in the space \( P^3 \), i.e. \( F^{(i)} \in P^3 \).

### 6.1 \( C^1 \)-smooth isogeometric functions

Subsequently, we use the tensor-product notation introduced in (2.3) for the trivariate case and let \( N_{\ell_1,\ell_2}^p = N_{\ell_1}^p N_{\ell_2}^p \), \( \ell_1,\ell_2 = 0,\ldots,p+k(p-1) \) be the bivariate tensor-product B-splines of the tensor-product spline space \( S_{1,k}^p \otimes S_{1,k}^p \).

We consider gluing data \( D_F \) given by

\[
\begin{align*}
\beta(u,v) &= \lambda \det \left( \partial_u F^{(1)}(u,v,0), \partial_u F^{(1)}(u,v,0), \partial_u F^{(2)}(u,v,0) \right), \\
\gamma(u,v) &= \lambda \det \left( \partial_u F^{(1)}(u,v,0), \partial_u F^{(1)}(u,v,0), \partial_u F^{(2)}(u,v,0) \right), \\
\alpha^{(2)}(u,v) &= \lambda \det \left( \partial_u F^{(2)}(u,v,0), \partial_u F^{(2)}(u,v,0), \partial_u F^{(2)}(u,v,0) \right), \\
\alpha^{(1)}(u,v) &= \lambda \det \left( \partial_u F^{(1)}(u,v,0), \partial_u F^{(1)}(u,v,0), \partial_u F^{(1)}(u,v,0) \right),
\end{align*}
\]

respectively, where \( \lambda \in \mathbb{R}_{>0} \) is determined by minimizing the term

\[
||\alpha^{(1)} + 1||^2_{L_2} + ||\alpha^{(2)} - 1||^2_{L_2}.
\]
Since $F^{(1)}$ and $F^{(2)}$ are trilinear, regular parameterizations, we know that this type of gluing data has degree $Q_3$, see (2.13), and it satisfies

$$\alpha^{(2)} > 0 \text{ and } \alpha^{(1)} < 0. \quad (6.1)$$

Furthermore, the two geometry mappings $F^{(1)}$ and $F^{(2)}$ fulfill

$$\beta \partial_u F^{(1)}|_{w=0} - \gamma \partial_v F^{(1)}|_{w=0} + \alpha^{(2)} \partial_w F^{(1)}|_{w=0} - \alpha^{(1)} \partial_w F^{(2)}|_{w=0} = 0, \quad (6.2)$$

since $F \in G^{3}_{D^3}$.

From the definition of the glued spline space (2.2) and Theorem 2.2, we know that an isogeometric function $\phi$ belongs to the space $\mathcal{V}_F$ if and only if the two corresponding functions $g^{(i)} = \phi \circ F^{(i)}$, $i \in \{1, 2\}$, satisfy

$$g^{(1)}|_{w=0} = g^{(2)}|_{w=0} = g_0 \quad (6.3)$$

and

$$\beta \partial_u g_0 - \gamma \partial_v g_0 + \alpha^{(2)} \partial_w g^{(1)}|_{w=0} - \alpha^{(1)} \partial_w g^{(2)}|_{w=0} = 0. \quad (6.4)$$

**Lemma 6.1.** There exist bivariate functions $\beta^{(i)}$, $\gamma^{(i)}$, $i \in \{1, 2\}$, such that

$$\beta = \beta^{(2)} \alpha^{(1)} - \beta^{(1)} \alpha^{(2)} \quad \text{and} \quad \gamma = -\gamma^{(2)} \alpha^{(1)} + \gamma^{(1)} \alpha^{(2)}. \quad (6.5)$$

**Proof.** One possible choice of the functions is

$$\beta^{(2)} = \frac{\beta}{2\alpha^{(1)}}, \quad \beta^{(1)} = -\frac{\beta}{2\alpha^{(2)}}, \quad \gamma^{(2)} = -\frac{\gamma}{2\alpha^{(1)}} \quad \text{and} \quad \gamma^{(1)} = \frac{\gamma}{2\alpha^{(2)}}. \quad (6.6)$$

\qed
Remark 6.1. The functions $\beta^{(i)}$ and $\gamma^{(i)}$ in (6.6) are well-defined due to (6.1), but they are not uniquely determined. For most configurations of the trilinearly parameterized subdomains $\Omega^{(1)}$ and $\Omega^{(2)}$ the functions $\beta^{(i)}$ and $\gamma^{(i)}$ can be selected e.g. as bilinear polynomials, see Section 6.2.

Lemma 6.1 allows us to rewrite equation (6.4):

Lemma 6.2. An isogeometric function $\phi$ belongs to the space $V_F$ if and only if the two corresponding functions $g^{(i)} = \phi \circ F^{(i)}$, $i \in \{1, 2\}$, satisfy equation (6.3) and

$$\frac{\partial_w g^{(1)}|_{w=0} - \beta^{(1)} \partial_u g_0 - \gamma^{(1)} \partial_v g_0}{\alpha^{(1)}} = \frac{\partial_w g^{(2)}|_{w=0} - \beta^{(2)} \partial_u g_0 - \gamma^{(2)} \partial_v g_0}{\alpha^{(2)}}, \quad (6.7)$$

Proof. It remains to show that equations (6.4) and (6.7) are equivalent. We can rewrite equation (6.4) with use of (6.5) and (6.3) as

$$\alpha^{(2)} \partial_w g^{(1)}|_{w=0} - \beta^{(1)} \alpha^{(2)} \partial_u g_0 - \gamma^{(1)} \alpha^{(2)} \partial_v g_0 = \alpha^{(1)} \partial_w g^{(2)}|_{w=0} - \beta^{(2)} \alpha^{(1)} \partial_u g_0 - \gamma^{(2)} \alpha^{(1)} \partial_v g_0.$$

A division by $\alpha^{(2)} \alpha^{(1)}$ on both sides of the equation yields equation (6.7). $\Box$

For each isogeometric function $\phi \in V_F$, the associated spline functions $g^{(i)} = \phi \circ F^{(i)}$, $i \in \{1, 2\}$, possess a spline representation of the form

$$\sum_{\ell_1=0}^{p+k(p-1)} \sum_{\ell_2=0}^{p+k(p-1)} \sum_{\ell_3=0}^{p+k(p-1)} d_{\ell_1,\ell_2,\ell_3}^{(i)} N_{\ell_1,\ell_2,\ell_3}^p(u,v,w), \quad d_{\ell_1,\ell_2,\ell_3}^{(i)} \in \mathbb{R}.$$

Recall, that we can split the space of $C^1$–smooth geometrically continuous isogeometric functions into the direct sum of the space of standard functions and interface
functions, see (3.3). The two subspaces have the following spline representations

\[ \mathcal{V}_F^S = \{ \phi \in \mathcal{V}_F : g^{(i)}(u, v, w) = (\phi \circ F^{(i)})(u, v, w) = \]

\[ \sum_{\ell_1=0}^{p+k(p-1)} \sum_{\ell_2=0}^{p+k(p-1)} \sum_{\ell_3=2}^{p+k(p-1)} d^{(i)}_{\ell_1,\ell_2,\ell_3} N^p_{\ell_1,\ell_2,\ell_3}(u, v, w), \quad d^{(i)}_{\ell_1,\ell_2,\ell_3} \in \mathbb{R}, i \in \{1, 2\} \} \]

and

\[ \mathcal{V}_F^F = \{ \phi \in \mathcal{V}_F : g^{(i)}(u, v, w) = (\phi \circ F^{(i)})(u, v, w) = \]

\[ \sum_{\ell_1=0}^{p+k(p-1)} \sum_{\ell_2=0}^{p+k(p-1)} \sum_{\ell_3=0}^{p+k(p-1)} d^{(i)}_{\ell_1,\ell_2,\ell_3} N^p_{\ell_1,\ell_2,\ell_3}(u, v, w), \quad d^{(i)}_{\ell_1,\ell_2,\ell_3} \in \mathbb{R}, i \in \{1, 2\} \}. \]

Furthermore, we obtain that the space \( \mathcal{V}_F^S \) can be simply described as

\[ \mathcal{V}_F^S = \{ \phi = (g^{(1)}, g^{(2)}) \circ F^{-1} : g^{(i)}(u, v, w) = \]

\[ \sum_{\ell_1=0}^{p+k(p-1)} \sum_{\ell_2=0}^{p+k(p-1)} \sum_{\ell_3=2}^{p+k(p-1)} d^{(i)}_{\ell_1,\ell_2,\ell_3} N^p_{\ell_1,\ell_2,\ell_3}(u, v, w), \quad d^{(i)}_{\ell_1,\ell_2,\ell_3} \in \mathbb{R}, i \in \{1, 2\} \}, \]

which implies that \( \dim \mathcal{V}_F^S \) is given by

\[ \dim \mathcal{V}_F^S = 2(p + 1 + k(p - 1))^2(p - 1 + k(p - 1)), \]

which is the same as in (3.4).

The collection of functions

\[ \{ \phi^{(i)}_{\ell_1,\ell_2,\ell_3} \}_{\ell_1,\ell_2=0,\ldots,p+k(p-1);\ell_3=2,\ldots,p+k(p-1);i\in\{1,2\}} \] (6.9)
with
\[
\phi^{(i)}_{\ell_1,\ell_2,\ell_3}(x) = \begin{cases} 
(N^p_{\ell_1,\ell_2,\ell_3} \circ (F^{(i)})^{-1})(x) & \text{if } x \in \Omega^{(i)}, \\
0 & \text{otherwise,}
\end{cases}
\]
form a basis of $V^S_F$, which are just the “standard” isogeometric basis functions.

To theoretically characterize the space $V^\Gamma_F$ we need the following lemma:

**Lemma 6.3.** Let $\phi \in V^\Gamma_F$. The functions $g^{(i)} = \phi \circ F^{(i)}$, $i \in \{1, 2\}$, can be represented as
\[
g^{(i)}(u,v,w) = g_0(u,v)\left(N^p_0(w) + N^p_1(w)\right) + \left(\beta^{(i)}(u,v)\partial_u g_0(u,v) + \gamma^{(i)}(u,v)\partial_v g_0(u,v) + \alpha^{(i)}(u,v)g_1(u,v)\right)\tau_1 p N^p_1(w),
\]
with $g_0$ as given in equation (6.3) and
\[
g_1 = \frac{\partial_w g^{(i)}|_{w=0} - \beta^{(1)}\partial_u g_0 - \gamma^{(1)}\partial_v g_0}{\alpha^{(1)}} = \frac{\partial_w g^{(2)}|_{w=0} - \beta^{(2)}\partial_u g_0 - \gamma^{(2)}\partial_v g_0}{\alpha^{(2)}}.
\]

**Proof.** Recall Lemma 6.2. We define functions $g_0, g_1 : [0,1]^2 \to \mathbb{R}$ as in equation (6.3) and (6.11), respectively. Then, the partial derivatives $\partial_w g^{(i)}|_{w=0}$, $i \in \{1, 2\}$, can be written as
\[
\partial_w g^{(i)}|_{w=0} = \beta^{(i)}\partial_u g_0 + \gamma^{(i)}\partial_v g_0 + \alpha^{(i)}g_1.
\]
Taylor approximation of $g^{(i)}$ at $(u,v,w) = (u,v,0)$ leads to
\[
g^{(i)}(u,v,w) = g^{(i)}(u,v,0) + \partial_w g^{(i)}(u,v,0)w + O(w^2) = g_0(u,v) + \left(\beta^{(i)}(u,v)\partial_u g_0(u,v) + \gamma^{(i)}(u,v)\partial_v g_0(u,v) + \alpha^{(i)}(u,v)g_1(u,v)\right)w + O(w^2),
\]
which is equal to (6.10) by using the fact that \( g^{(i)} \) is a spline function with a representation shown in (6.8).

We obtain:

**Theorem 6.4.** The space \( \mathcal{V}_F^\Gamma \) is equal to

\[
\mathcal{V}_F^\Gamma = \left\{ \phi = (g^{(1)}, g^{(2)}) \circ F^{-1} : g^{(i)} \text{ is given by (6.10), } g_0, g_1 : [0, 1]^2 \to \mathbb{R} \text{ such that } g^{(i)} \in \mathcal{P}, \ i \in \{1, 2\} \right\}.
\]

**Proof.** On the one hand if \( \phi \in \mathcal{V}_F^\Gamma \), then clearly \( g^{(i)} = \phi \circ F^{(i)} \in \mathcal{P}, \ i \in \{1, 2\} \), and Lemma 6.3 ensures that the functions \( g^{(i)} \) possess a representation (6.10) with functions \( g_0, g_1 : [0, 1]^2 \to \mathbb{R} \). On the other hand if the functions \( g^{(i)}, \ i \in \{1, 2\} \), possess a representation (6.10) then conditions (6.3) and (6.7) are fulfilled which implies by means of Lemma 6.2 and the fact \( g^{(i)} \in \mathcal{P} \) that \( \phi = (g^{(1)}, g^{(2)}) \circ F^{-1} \in \mathcal{V}_F^\Gamma \). Since \( g^{(i)} \) possess a representation of the form shown in (6.8) we also obtain that \( \phi \in \mathcal{V}_F^\Gamma \).

Furthermore, the two functions \( g_0 \) and \( g_1 \) in (6.10) have nice geometric interpretations. In order to derive them, we use a vector \( \mathbf{d} \), which is transversal to \( \Gamma \). This vector was already introduced in [15] for the case of planar two-patch domains and can be extended in a straightforward way to the case of volumetric two-patch domains. Recall (2.1), then we define the vector \( \mathbf{d} = \mathbf{d}^{(1)} = \mathbf{d}^{(2)} \) on \( \Gamma \) as

\[
\mathbf{d}^{(i)} \circ F_0 = \left( \partial_u F_0, \partial_v F_0, \partial_w F^{(i)}|_{w=0} \right) \cdot \left( -\beta^{(i)}, -\gamma^{(i)}, 1 \right) \frac{1}{\alpha^{(i)}}, \ i \in \{1, 2\}.
\]

The vector \( \mathbf{d} \) is well-defined, i.e. \( \mathbf{d}^{(1)} = \mathbf{d}^{(2)} \), since the equation

\[
\alpha^{(1)} \alpha^{(2)} \mathbf{d}^{(1)} \circ F_0 = \alpha^{(1)} \alpha^{(2)} \mathbf{d}^{(2)} \circ F_0 = 0
\]
is equivalent to the system of equations (6.2) and (6.5). Furthermore, \( d \) is transversal to \( \Gamma \), since

\[
\det \left( \partial_u F_0, \partial_v F_0, d^{(i)} \circ F_0 \right) = \frac{1}{\alpha^{(i)}} \det \left( \partial_u F_0, \partial_v F_0, \partial_w F_0 \big|_{w=0} \right) = \frac{1}{\lambda} \neq 0.
\]

Let us consider the directional derivative of \( \phi \in \mathcal{V}_F^1 \) from side \( i \) with respect to \( d \) at \( \Gamma \), i.e. \( \nabla^{(i)} \phi \cdot d \circ F_0 \), which is equal to

\[
\left( \partial_u g^{(i)} \big|_{w=0}, \partial_v g^{(i)} \big|_{w=0}, \partial_w g^{(i)} \big|_{w=0} \right) \cdot \left( \partial_u F_0, \partial_v F_0, \partial_w F_0 \big|_{w=0} \right)^{-1} \cdot d \circ F_0 = \left( \partial_u g^{(i)} \big|_{w=0}, \partial_v g^{(i)} \big|_{w=0}, \partial_w g^{(i)} \big|_{w=0} \right) \cdot \left( -\beta^{(i)}, -\gamma^{(i)}, 1 \right)^T \frac{1}{\alpha^{(i)}} \frac{\partial_w g^{(i)} \big|_{w=0} - \beta^{(i)} \partial_u g^{(i)} \big|_{w=0} - \gamma^{(i)} \partial_v g^{(i)} \big|_{w=0}}{\alpha^{(i)}}.
\]

Since \( \phi \) is \( C^1 \)-smooth on \( \Omega \), equation (6.7) is satisfied and therefore we get

\[
\nabla^{(1)} \phi \cdot d \circ F_0 = \nabla^{(2)} \phi \cdot d \circ F_0 = \nabla \phi \cdot d \circ F_0.
\]

It follows:

**Corollary 6.5.** Let \( \phi \in \mathcal{V}_F^1 \), and consider the associated representation (6.10) from the function \( g^{(i)} = \phi \circ F^{(i)}, i \in \{1, 2\} \). The function \( g_0 \) is the trace of the function \( \phi \) at \( \Gamma \), i.e.

\[
g_0 = \phi \circ F_0,
\]

and the function \( g_1 \) is the directional derivative of the function \( \phi \) with respect to the transversal direction \( d \) at \( \Gamma \), i.e.

\[
g_1 = \nabla \phi \cdot d \circ F_0.
\]
6.2 The generic case

Below, we restrict ourselves to the generic case of trilinearly parameterized two-patch domains $\Omega$ introduced in Section 2.3, which can be seen as Type $T_{\text{trl}}$ valid with probability 1. For this we assume that

- the common interface $\Gamma$ is nonplanar,
- the functions $\beta$, $\gamma$, $\alpha^{(2)}$ and $\alpha^{(1)}$ possess full bidegrees $(3, 2)$, $(2, 3)$, $(2, 2)$ and $(2, 2)$, respectively,
- the functions $\beta$ and $\gamma$ do not have roots at the values $(\tau_i, \tau_j)$, $i, j = 1, \ldots, k$, where $\tau_i$ and $\tau_j$ are the inner knots of the spline space $S^p_k$, compare (1.1), and that
- the greatest common divisor of the two functions $\alpha^{(2)}$ and $\alpha^{(1)}$ is a constant function.

Note that any trilinearly parameterized two-patch domain, which does not satisfy these assumptions, can be transformed into one fulfilling the assumptions by just slightly disturbing some of the values of the vertices $c_{j,k,\ell}$.

In the following, let $\Pi^{(q_1,q_2)}$ and $\Pi^{(q_1,q_2,q_3)}$ be the space of bivariate and trivariate polynomials on $[0,1]^2$ and $[0,1]^3$ of bidegree $(q_1,q_2) \in \mathbb{Z}_{\geq 0}^2$ and tridegree $(q_1,q_2,q_3) \in \mathbb{Z}_{\geq 0}^3$, respectively. By investigating the functions $\beta$, $\gamma$, $\alpha^{(2)}$ and $\alpha^{(1)}$, we obtain:

**Lemma 6.6.** There exist bilinear polynomials $\beta^{(i)}$, $\gamma^{(i)}$, $i \in \{1, 2\}$ (i.e. $\beta^{(i)}$ and $\gamma^{(i)} \in \Pi^{(1,1)}$) such that

$$\beta = \beta^{(2)} \alpha^{(1)} - \beta^{(1)} \alpha^{(2)} \quad \text{and} \quad \gamma = -\gamma^{(2)} \alpha^{(1)} + \gamma^{(1)} \alpha^{(2)}, \quad (6.12)$$
with

\[
\beta^{(i)}(u, v) = \frac{\det \left( c_{1,0,0} - c_{1,0,0,0} - c_{0,1,0,0} - c_{0,0,0,0} \right)}{\text{vol}} \left. \partial_w F^{(i)} \right|_{w=0}
\]

and

\[
\gamma^{(i)}(u, v) = \frac{\det \left( c_{1,1,0} - c_{0,1,0,0} - c_{0,0,0} - c_{0,0,0} \right)}{\text{vol}} \left. \partial_w F^{(i)} \right|_{w=0},
\]

where

\[
\text{vol} = \det \left( c_{1,0,0} - c_{0,0,0} - c_{0,0,0} - c_{1,1,0} - c_{0,0,0} \right).
\]

(6.13)

In addition, the biquadratic function \( \alpha^{(i)} \in \Pi^{(2,2)}, i \in \{1, 2\} \), possesses the form

\[
\alpha^{(i)}(u, v) = \lambda \cdot \text{vol} \left( \eta^{(i)}(u, v) - u \gamma^{(i)}(u, v) - v \beta^{(i)}(u, v) \right),
\]

(6.15)

where \( \eta^{(i)}, i \in \{1, 2\} \), is a bilinear function (i.e. \( \eta^{(i)} \in \Pi^{(1,1)} \)) given by

\[
\eta^{(i)}(u, v) = \frac{\det \left( c_{1,0,0} - c_{0,0,0} - c_{0,0,0} - c_{0,0,0} \right)}{\text{vol}} \left. \partial_w F^{(i)} \right|_{w=0}.
\]

(6.14)

Proof. We first show equation (6.15). Since \( F^{(1)} \) and \( F^{(2)} \) are trilinear parameterizations which are determined by the vertices \( c_{j,k,l} \) as shown in Figure 2.2, we have that

\[
\partial_u F_0 = (c_{1,0,0} - c_{0,0,0}) + v(c_{0,0,0} - c_{1,0,0} - c_{0,1,0} + c_{1,1,0}), \quad i \in \{1, 2\},
\]

and

\[
\partial_v F_0 = (c_{0,1,0} - c_{0,0,0}) + u(c_{0,0,0} - c_{1,0,0} - c_{0,1,0} + c_{1,1,0}), \quad i \in \{1, 2\}.
\]

Using this fact and some basic properties of determinants, one can easily verify that
The generic case

\[ \alpha^{(i)}(u, v) = \lambda \det \left( \partial_u F_0, \partial_2 F_0, \partial_3 F^{(i)} |_{u=0} \right) = \]
\[ \lambda \left( \det \left( c_{1,0,0} - c_{0,0,0}, c_{0,1,0} - c_{0,0,0}, \partial_3 F^{(i)} |_{u=0} \right) - \right. \]
\[ u \det \left( c_{1,1,0} - c_{0,1,0}, c_{0,1,0} - c_{0,0,0}, \partial_3 F^{(i)} |_{u=0} \right) - \]
\[ v \det \left( c_{1,1,0} - c_{1,0,0}, c_{0,0,0} - c_{1,1,0}, \partial_3 F^{(i)} |_{u=0} \right) \]

which is equivalent to (6.15).

It remains to prove that equation (6.12) is satisfied by selecting functions \( \beta^{(i)} \) and \( \gamma^{(i)} \), \( i \in \{1, 2\} \), as in (6.13). We will show it only for the case of \( \beta \), since the case of \( \gamma \) would work analogously. In addition, we will omit the arguments of the functions for the sake of brevity. We have

\[ \beta^{(2)} \alpha^{(1)} - \beta^{(1)} \alpha^{(2)} = \]
\[ \lambda \cdot \text{vol} \left( \eta^{(1)} - u \gamma^{(1)} - v \beta^{(1)} \right) \beta^{(2)} - \lambda \cdot \text{vol} \left( \eta^{(2)} - w \gamma^{(2)} - v \beta^{(2)} \right) \beta^{(1)} = \]
\[ \lambda \cdot \text{vol} \left( \beta^{(2)} \eta^{(1)} - u \beta^{(2)} \gamma^{(1)} - \beta^{(1)} \eta^{(2)} + u \beta^{(1)} \gamma^{(2)} \right). \quad (6.16) \]

By substituting the vectors \( c_{1,0,0} - c_{0,0,0}, c_{0,1,0} - c_{0,0,0} \) and \( c_{1,1,0} - c_{0,0,0} \) by \( a, b \) and \( c \), respectively, the term (6.16) is equal to

\[ \frac{1}{\det(a, b, c)} \lambda \cdot \]
\[ \left( \det(b, c, \partial_3 F^{(2)} |_{w=0}) \det(a, b, \partial_3 F^{(1)} |_{w=0}) - \det(b, c, \partial_3 F^{(1)} |_{w=0}) \det(a, b, \partial_3 F^{(2)} |_{w=0}) + \right. \]
\[ u \left( \det(b, c, \partial_3 F^{(1)} |_{w=0}) \det(a, b, \partial_3 F^{(1)} |_{w=0}) + \det(b, c, \partial_3 F^{(1)} |_{w=0}) \det(c, a, \partial_3 F^{(2)} |_{w=0}) + \right. \]
\[ \det(a, b, \partial_3 F^{(1)} |_{w=0}) \det(c, a, \partial_3 F^{(2)} |_{w=0}) - \det(b, c, \partial_3 F^{(2)} |_{w=0}) \det(c, a, \partial_3 F^{(2)} |_{w=0}) \right) \].
Since the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent, which is a direct consequence of the fact that the interface $\Gamma$ is non-planar, the partial derivatives $\partial_3 \mathbf{F}(1)|_{w=0}$ and $\partial_3 \mathbf{F}(2)|_{w=0}$ can be written as

$$\partial_3 \mathbf{F}(i)|_{w=0} = \eta^{(i)} \mathbf{a} + \theta^{(i)} \mathbf{b} + \omega^{(i)} \mathbf{c}, \quad i \in \{1, 2\}. \quad (6.17)$$

with bivariate functions $\eta^{(i)}$, $\theta^{(i)}$ and $\omega^{(i)}$, $i \in \{1, 2\}$. Then, the previous term is equal to

$$\lambda \left( \frac{\eta^{(2)} \omega^{(1)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 - \eta^{(1)} \omega^{(2)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2}{\det(\mathbf{a}, \mathbf{b}, \mathbf{c})} + u \left( \eta^{(1)} \omega^{(2)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 + \eta^{(2)} \theta^{(1)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 + \omega^{(1)} \theta^{(2)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 - \eta^{(2)} \omega^{(1)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 - \eta^{(1)} \theta^{(2)} \det(\mathbf{a}, \mathbf{b}, \mathbf{c})^2 \right) \right).$$

Using again relation (6.17), this can be expressed as

$$\lambda \left( \det(\partial_3 \mathbf{F}(2)|_{w=0}, \mathbf{b}, \partial_3 \mathbf{F}(1)|_{w=0}) + u \left( -\det(\mathbf{a}, \partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(2)|_{w=0}) + \det(\partial_3 \mathbf{F}(1)|_{w=0}, \mathbf{b}, \partial_3 \mathbf{F}(2)|_{w=0}) \right) + \det(\partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(2)|_{w=0}) \right) =$$

$$\lambda \left( \det(\partial_3 \mathbf{F}(2)|_{w=0}, \mathbf{b}, \partial_3 \mathbf{F}(1)|_{w=0}) + u \left( \det(-\mathbf{a} + \mathbf{b}, \partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(2)|_{w=0}) \right) \right).$$

By resubstituting $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, this term can be written as

$$\lambda \det((\mathbf{c}_{0,1,0} - \mathbf{c}_{0,0,0}) + u((\mathbf{c}_{0,0,0} - \mathbf{c}_{1,0,0} - \mathbf{c}_{0,1,0} + \mathbf{c}_{1,1,0}), \partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(2)|_{w=0}) = \lambda \det(\partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(1)|_{w=0}, \partial_3 \mathbf{F}(2)|_{w=0}),$$

which is exactly the function $\beta$. \qed
Remark 6.2. The value $\text{vol}$ defined in (6.14) is the volume of the rectangular solid spanned by the three vectors $c_{1,0,0} - c_{0,0,0}, c_{0,1,0} - c_{0,0,0}$ and $c_{1,1,0} - c_{0,0,0}$, where $c_{0,0,0}, c_{1,0,0}, c_{0,1,0}$ and $c_{1,1,0}$ are the four vertices of the common interface, see Figure 2.2.

Before we can study the space $\mathcal{V}_F^\Gamma$ in more detail and to construct a basis, we need some additional functions. Let $R_{\ell}^{p,r+1} : [0, 1] \rightarrow \mathbb{R}, i = 0, \ldots, p + k(p - r - 1)$, be the function defined by

$$R_{\ell}^{p,r+1}(x) = \frac{1}{t_{\ell+p}-t_{\ell}} N_{p-1,r}^{\ell,1}(x) - \frac{1}{t_{\ell+p+1}-t_{\ell+1}} N_{p-1,r}^{\ell,2}(x),$$

(6.18)

where $t_{\ell}$ are the knots with respect to the spline space $S_{\ell}^{p,r+1}$. Note that we denote the regularity of the splines by $r$, in specific we consider $r = 1$. Hereafter, we set the term $\frac{1}{t_{\ell+p}-t_{\ell}}$ or $\frac{1}{t_{\ell+p+1}-t_{\ell+1}}$ to be zero, if the corresponding denominator is equal to zero. For these cases, which only happen for $i = 0$ for the first term and $i = p + k(p - r - 1)$ for the second term, we also assume that the B-splines $N_{p-1,r}^{\ell,1}$ and $N_{p+k(p-r-1)}^{p,r-1}$ are given as zero functions.

Moreover, we define the functions $\phi_{\ell_1,\ell_2}^{\Gamma,0}, \ell_1, \ell_2 \in \{0, \ldots, p + k(p - r - 1)\}$ and $\phi_{j_1,j_2}^{\Gamma,1}, j_1, j_2 \in \{0, \ldots, p + 2 + k(p - r - 2)\}$, as the isogeometric functions

$$((g^{(1)}, g^{(2)}) \circ F^{-1})(x)$$

with $g^{(i)}, i \in \{1, 2\}$, given in (6.10), and

$$g_0(u, v) = N_{\ell_1,\ell_2}^{p,r+1}(u, v) \quad \text{and} \quad g_1(u, v) = \frac{p}{\lambda \cdot \text{vol}_R} R_{\ell_1}^{p,r+1}(u) R_{\ell_2}^{p,r+1}(v)$$

(6.19)

for $\phi_{\ell_1,\ell_2}^{\Gamma,0}$, and

$$g_0(u, v) = 0 \quad \text{and} \quad g_1(u, v) = N_{j_1,j_2}^{p-2,r}(u, v)$$

(6.20)
for $\phi_{j_1,j_2}^{\Gamma,1}$.

Clearly, the supports of the functions $\phi_{\ell_1,\ell_2}^{\Gamma,0}$ and $\phi_{j_1,j_2}^{\Gamma,1}$ are inherited by the B-splines $N_{\ell}^{p-1,r}$, $N_{\ell_1,\ell_2}^{p,r+1}$ and $N_{j_1,j_2}^{p-2,r}$ used in (6.18), (6.19) and (6.20), respectively. Therefore, except for $r = p - 1$ in case of $\phi_{\ell_1,\ell_2}^{\Gamma,0}$ and for $p - 2 \leq r \leq p - 1$ in case of $\phi_{j_1,j_2}^{\Gamma,1}$, the functions $\phi_{\ell_1,\ell_2}^{\Gamma,0}$ and $\phi_{j_1,j_2}^{\Gamma,1}$ possess a small local support. For the excluded cases, the B-splines $N_{\ell}^{p-1,r}$, $N_{\ell_1,\ell_2}^{p,r+1}$ and $N_{j_1,j_2}^{p-2,r}$ are just Bernstein polynomials defined on the interval $[0, 1]$ or on the unit square $[0, 1]^2$, and hence the resulting functions $\phi_{\ell_1,\ell_2}^{\Gamma,0}$ and $\phi_{j_1,j_2}^{\Gamma,1}$ possess independent of the selected number $k$ of inner knots a support over the entire interface $\Gamma$.

**Lemma 6.7.** The functions $\phi_{\ell_1,\ell_2}^{\Gamma,0}$, $\ell_1, \ell_2 \in \{0, \ldots, p + k(p - r - 1)\}$ and $\phi_{j_1,j_2}^{\Gamma,1}$, $j_1, j_2 \in \{0, \ldots, p - 2 + k(p - r - 2)\}$, belong to the space $\mathcal{V}_F$.

**Proof.** Thanks to Theorem 6.4, it is sufficient to prove that

$$
\phi_{\ell_1,\ell_2}^{\Gamma,0} \circ F^{(i)}, \phi_{j_1,j_2}^{\Gamma,1} \circ F^{(i)} \in \mathcal{P}, \ i \in \{1, 2\}.
$$

In case of $\phi_{j_1,j_2}^{\Gamma,1}$, this can be easily seen by considering the representation (6.10) of the function $g^{(i)} = \phi_{j_1,j_2}^{\Gamma,1} \circ F^{(i)}$. In case of $\phi_{\ell_1,\ell_2}^{\Gamma,0}$, it remains to show that the second term of the representation (6.10) of the function $g^{(i)} = \phi_{\ell_1,\ell_2}^{\Gamma,0} \circ F^{(i)}$, i.e.

$$
(\beta^{(i)}(u, v) \partial_u g_0(u, v) + \gamma^{(i)}(u, v) \partial_v g_0(u, v) + \alpha^{(i)}(u, v) g_1(u, v)) \frac{\tau_1}{p} N_1^{p,r}(w)
$$

belongs to the space $\mathcal{P}$, or simply

$$
\beta^{(i)}(u, v) \partial_u g_0(u, v) + \gamma^{(i)}(u, v) \partial_v g_0(u, v) + \alpha^{(i)}(u, v) g_1(u, v) \in S_{r,k}^p \otimes S_{r,k}^p,
$$

(6.21)

since the first term of (6.10) trivially belongs to $\mathcal{P}$.

Let us first recall the well-known recursion formula

$$
N_{\ell}^{p,r+1}(x) = \frac{x - t_\ell}{t_{\ell+p} - t_\ell} N_{\ell}^{p-1,r}(x) + \frac{t_{\ell+p+1} - x}{t_{\ell+p+1} - t_{\ell+1}} N_{\ell}^{p-1,r}(x),
$$

(6.22)
and the well-known derivative relation

$$
\left(N^{p,r+1}_t(x)\right)' = p \left(\frac{1}{t_{\ell+p} - t_{\ell}} N^{p-1,r}_{t_{\ell}-1}(x) - \frac{1}{t_{\ell+p+1} - t_{\ell+1}} N^{p-1,r}_{t_{\ell+1}}(x) \right). \quad (6.23)
$$

By means of Lemma 6.6 and (6.19), we get

$$
\beta^{(i)}(u,v) \partial_u g_0(u,v) + \gamma^{(i)}(u,v) \partial_v g_0(u,v) + \alpha^{(i)}(u,v) g_1(u,v) =
\beta^{(i)}(u,v) \left(N^{p,r+1}_t(u)\right)' \ N^{p,r+1}_t(v) + \gamma^{(i)}(u,v) N^{p,r+1}_t(u) \left(N^{p,r+1}_t(v)\right)' +
\rho \eta^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) - u \gamma^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) -
u \beta^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v).
$$

Equation (6.18) and relation (6.23) lead to

$$
\beta^{(i)}(u,v) \left(N^{p,r+1}_t(u)\right)' \ N^{p,r+1}_t(v) + \gamma^{(i)}(u,v) N^{p,r+1}_t(u) \left(N^{p,r+1}_t(v)\right)' +
\rho \eta^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) - u \gamma^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) -
u \beta^{(i)}(u,v) \left(N^{p,r+1}_t(u)\right)' R^{p,r+1}_{t_2}(v).
$$

Using again equation (6.18) and the fact that the linear function $x$ can be rewritten as $x = x - c + c$ for all constants $c \in \mathbb{R}$, this is equal to

$$
\beta^{(i)}(u,v) \left(N^{p,r+1}_t(u)\right)' \ N^{p,r+1}_t(v) + \gamma^{(i)}(u,v) N^{p,r+1}_t(u) \left(N^{p,r+1}_t(v)\right)' +
\rho \eta^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) - u \gamma^{(i)}(u,v) R^{p,r+1}_{t_1}(u) R^{p,r+1}_{t_2}(v) -
u \beta^{(i)}(u,v) \left(N^{p,r+1}_t(u)\right)' R^{p,r+1}_{t_2}(v).
$$
With the help of the recurrence relation (6.22), this can be simplified to

\[
p n^{(i)}(u,v)N_{\ell_1}^{p,r+1}(u)N_{\ell_2}^{p,r+1}(v) - \gamma^{(i)}(u,v) \left( \frac{t_{\ell_1}}{t_{\ell_1+p} - t_{\ell_1}} N_{\ell_1-1}^{p-1,r}(u) - \frac{t_{\ell_1+p+1}}{t_{\ell_1+p+1} - t_{\ell_1}} N_{\ell_1}^{p-1,r}(u) \right) \left( N_{\ell_2}^{p,r+1}(v) \right)' - \beta^{(i)}(u,v) \left( N_{\ell_1}^{p,r+1}(u) \right)'.
\]

which finally implies condition (6.21).

\[\square\]

6.3 Dimension and basis of the space \(V^\Gamma_F\)

We have that \(\phi_{\ell_1,\ell_2}^{\Gamma,0}, \phi_{ji,j_2}^{\Gamma,1} \in V^\Gamma_F\), compare Lemma 6.7, and note that the set of functions \(\phi_{\ell_1,\ell_2}^{\Gamma,0}\) and \(\phi_{ji,j_2}^{\Gamma,1}\) are linearly independent by definition. In order to obtain a basis of the space \(V^\Gamma_F\) we still require that each function \(\phi \in V^\Gamma_F\) can be represented as a linear combination of the functions \(\phi_{\ell_1,\ell_2}^{\Gamma,0}\) and \(\phi_{ji,j_2}^{\Gamma,1}\). Then we have:

**Theorem 6.8.** A basis of the space \(V^\Gamma_F\) is given by the collection of functions

\[
\{\phi_{\ell_1,\ell_2}^{\Gamma,0}, \phi_{ji,j_2}^{\Gamma,1}\}_{\ell_1,\ell_2 \in \{0, ..., p+k(p-r-1)\}; j_1,j_2 \in \{0, ..., p-2+k(p-r-2)\}}.
\]

**Proof.** For this purpose, we first show that for an isogeometric function \(\phi \in V^\Gamma_F\) the function \(g_0\) from the representation (6.10) has to belong to the space \(S_k^{p,r+1} \otimes S_k^{p,r+1}\), and if \(g_0 = 0\), then the function \(g_1\) has to belong to the space \(S_k^{p-2,r} \otimes S_k^{p-2,r}\).

Let \(\phi \in V^\Gamma_F\). Note that the functions \(g^{(i)} = \phi \circ F^{(i)}\), \(i \in \{1,2\}\), have to fulfill equation (6.4), and it holds that \(g_0 = g^{(1)}|_{w=0} = g^{(2)}|_{w=0}\), compare (6.3). Since the functions \(\beta\) and \(\gamma\) do not have roots at the values \((\tau_\ell, \tau_j)\), \(i,j = 1, \ldots, k\), see the assumptions in the beginning of this section, we obtain that \(g_0 \in S_k^{p,r+1} \otimes S_k^{p,r+1}\).
Moreover, when $g_0 = 0$ in the representation (6.10), then equation (6.11) simplifies to

$$g_1 = \frac{\partial_w g^{(1)}|_{w=0}}{\alpha^{(1)}} = \frac{\partial_w g^{(2)}|_{w=0}}{\alpha^{(2)}}.$$

Since $\partial_w g^{(1)}|_{w=0}, \partial_w g^{(2)}|_{w=0} \in S^{p,r}_k \otimes S^{p,r}_k$ and hence $\frac{\alpha^{(2)}\partial_w g^{(1)}|_{w=0}}{\alpha^{(2)}}$ and $\frac{\alpha^{(1)}\partial_w g^{(2)}|_{w=0}}{\alpha^{(1)}} \in S^{p,r}_k \otimes S^{p,r}_k$, we obtain $g_1 \in S^{p-2,r}_k \otimes S^{p-2,r}_k$ by using the fact that $\alpha^{(1)}$ and $\alpha^{(2)}$ are biquadratic polynomials with a greatest common divisor of bidegree $(0,0)$.

Now, we are ready to show that each function $\phi \in \mathcal{V}_k^\Gamma$ can be written as a linear combination

$$\phi(x) = \sum_{\ell_1=0}^{p+k(p-r-1)} \sum_{\ell_2=0}^{p+k(p-r-1)} \mu_{\ell_1,\ell_2}^0 \phi_{\ell_1,\ell_2}^\Gamma(x) + \sum_{j_1=0}^{p-2+k(p-r-2)} \sum_{j_2=0}^{p-2+k(p-r-2)} \mu_{j_1,j_2}^1 \phi_{j_1,j_2}^\Gamma(x)$$

with factors $\mu_{\ell_1,\ell_2}^0, \mu_{j_1,j_2}^1 \in \mathbb{R}$. Let $\phi \in \mathcal{V}_k^\Gamma$ and consider the associated representation (6.10) from the function $g^{(i)} = \phi \circ F^{(i)}, i \in \{1,2\}$. Since $g_0 \in S^{p,r+1}_k \otimes S^{p,r+1}_k$, the function $g_0$ can be written as

$$g_0(u, v) = \sum_{\ell_1=0}^{p+k(p-r-1)} \sum_{\ell_2=0}^{p+k(p-r-1)} \mu_{\ell_1,\ell_2}^0 N_{p,r+1}^{\ell_1,\ell_2}(u, v)$$

with factors $\mu_{\ell_1,\ell_2}^0 \in \mathbb{R}$. Computing

$$\tilde{\phi}(x) = \phi(x) - \sum_{\ell_1=0}^{p+k(p-r-1)} \sum_{\ell_2=0}^{p+k(p-r-1)} \mu_{\ell_1,\ell_2}^0 \phi_{\ell_1,\ell_2}^0(x)$$

leads to a function $\tilde{\phi} \in \mathcal{V}_k^\Gamma$, whose spline functions $\tilde{g}^{(i)} = \tilde{\phi} \circ F^{(i)}, i \in \{1,2\}$, possess a representation (6.10) with $\tilde{g}_0 = 0$ and $\tilde{g}_1 \in S^{p-2,r}_k \otimes S^{p-2,r}_k$. Let $\mu_{j_1,j_2}^1 \in \mathbb{R}$ be the factors such that
\( \tilde{g}_1(u, v) = \sum_{j_1=0}^{p-r+2} \sum_{j_2=0}^{p-r+2} \mu_{j_1,j_2}^1 N^{p-r}_j u, v \),

then we have that

\[ \tilde{\phi}(x) - \sum_{j_1=0}^{p-r+2} \sum_{j_2=0}^{p-r+2} \mu_{j_1,j_2}^1 \phi_{j_1,j_2}^{1,1}(x) = 0. \]

This implies that \( \phi \) can be written as a linear combination (6.25) with the factors \( \mu_{\ell_1,\ell_2}^{\Gamma,0} = \mu_{\ell_1,\ell_2}^0 \) and \( \mu_{j_1,j_2}^{\Gamma,1} = \mu_{j_1,j_2}^1 \).

**Remark 6.3.** Bivariate \( C^1 \)-smooth isogeometric spline spaces over the class of analysis-suitable \( G^1 \) two-patch parameterizations (which also includes the subclass of bilinearly parameterized two-patch domains) enjoy optimal approximation properties, see [15]. This observation was based on the fact that the traces and transversal directional derivatives along the common interface can be chosen independently from a univariate spline space of degree \( p \) and of degree \( p - 1 \), respectively.

In contrast, in our case for the trivariate \( C^1 \)-smooth space \( \mathcal{V}_F \), the traces (i.e. the function \( g_0 \)) and the transversal directional derivatives (i.e. the function \( g_1 \)) along the interface \( \Gamma \) can be only selected independently from a tensor-product spline space of bidegree \( (p-r, p-r) \) and of bidegree \( (p-1, p-1) \), respectively. This combined with the constraints on the regularity explains the results from Section 5. The restricted choice of the transversal directional derivatives \( g_1 \) was already demonstrated in the proof of Theorem 6.8 by showing that \( g_0 = 0 \) implies \( g_1 \in S_k^{p-r,2} \otimes S_k^{p-r,2} \). Similarly, it could be shown for the traces \( g_0 \) that \( g_1 = 0 \) would lead to \( g_0 \in S_k^{p-1,r+1} \otimes S_k^{p-1,r+1} \).

We directly obtain the following two corollaries:

**Corollary 6.9.** The collections of functions (6.9) and (6.24) form a basis of the space \( \mathcal{V}_F \).
Note that in the following we set $r = 1$, since we consider $C^1$–smooth isogeometric functions.

**Corollary 6.10.** It holds that

$$\dim V_\Gamma^F = (p - 1 + k(p - 3))^2 + (p + 1 + k(p - 2))^2,$$

and hence

$$\dim V_F = 2(p + 1 + k(p - 1))^2(p - 1 + k(p - 1)) + \dim V_\Gamma^F \left( (p - 1 + k(p - 3))^2 + (p + 1 + k(p - 2))^2 \right).$$

**Remark 6.4.** The dimension results match with the ones in Chapter 3, where they were numerically shown, i.e.

$$\dim V_\Gamma^F = 2 + 2k + 13k^2 - 10k(1 + k)p + 2(k + 1)^2p^2.$$  

There, it was also demonstrated that the dimension does not change when the generic case of a planar interface is considered.

**Example 6.1.** We consider the generic trilinearly parameterized two-patch domain $\Omega$ visualized in Figure 6.1, which is defined by the 12 vertices

$$c_{0,0,1} = \left(0, 0, \frac{-6}{5}\right), \quad c_{1,0,-1} = \left(\frac{10}{9} \cdot \frac{1}{10}, -\frac{32}{31}\right), \quad c_{0,1,-1} = \left(\frac{1}{25}, \frac{16}{15}, -\frac{19}{20}\right),$$

$$c_{1,1,-1} = \left(\frac{16}{15}, \frac{14}{15}, -\frac{9}{10}\right), \quad c_{0,0,0} = \left(\frac{1}{17}, \frac{1}{20}, \frac{2}{15}\right), \quad c_{1,0,0} = \left(\frac{13}{15}, \frac{1}{15}, \frac{2}{25}\right),$$

$$c_{0,1,0} = \left(\frac{1}{12}, \frac{14}{15}, \frac{1}{15}\right), \quad c_{1,1,0} = \left(\frac{18}{19}, \frac{17}{18}, \frac{2}{25}\right), \quad c_{0,0,1} = \left(\frac{1}{10}, \frac{1}{15}, \frac{33}{35}\right).$$
$$c_{1,0,1} = \left(\frac{16}{15}, \frac{1}{13}, \frac{51}{50}\right), \quad c_{0,1,1} = \left(\frac{1}{15}, \frac{14}{15}, \frac{34}{35}\right), \quad c_{1,1,1} = \left(\frac{26}{25}, \frac{34}{35}, \frac{21}{20}\right).$$

According to Lemma 6.6, we obtain

$$\text{vol} = \frac{1086607}{21802500},$$

$$\beta^{(1)}(u, v) = \frac{93(33064760 - 8289039 v) + u(306334976 + 105798337 v)}{2425306824},$$

$$\beta^{(2)}(u, v) = \frac{-260(6954233 + 1471621 v) + u(134969134 + 8058829 v)}{2373149688},$$

$$\gamma^{(1)}(u, v) = \frac{u(2768879480 - 6732851123 v) + 93(-52469680 + 63462141 v)}{39168705207600},$$

$$\gamma^{(2)}(u, v) = \frac{-260(-9716185 + 702889 v) + u(101861470 + 754743721 v)}{114979102383600},$$

$$\alpha^{(1)}(u, v) = \frac{u^2(-2768879480 + 6732851123 v) - 10 u(-9052554324 + 8193890948 v + 105798337 v^2) + 93(-5003665200 + 905234209 v + 82890390 v^2)}{486631800000},$$

$$\alpha^{(2)}(u, v) = \frac{-u^2(101861470 + 754743721 v) + u(45101336050 - 23570690651 v - 80588290 v^2) + 260(1062421461 + 187459313 v + 14716210 v^2)}{476166600000}. $$
We choose for the degree \( p = 4 \) and for the number of inner knots \( k = 3 \). Due to Corollary 6.10, we get

\[
\dim \mathcal{V}_F = \dim \mathcal{V}^S_F + \dim \mathcal{V}^\Gamma_F = 4704 + 157,
\]

which is in accordance with Table 3.2. Theorem 6.8 provides us a basis for the resulting space \( \mathcal{V}^\Gamma_F \), which is given by the functions

\[
\{ \phi_{\ell_1, \ell_2}^{\Gamma,0,}, \phi_{j_1, j_2}^{\Gamma,1} \}_{\ell_1, \ell_2 \in \{0, \ldots, 10\}; j_1, j_2 \in \{0, \ldots, 5\}}.
\]

Figure 6.2 illustrates two instances of basis functions, namely the functions \( \phi_{5,5}^{\Gamma,0} \) and \( \phi_{2,2}^{\Gamma,1} \), and show that the basis functions possess a small local support.
Finally, based on a concrete example, we also present one possible strategy to construct $C^1$–smooth isogeometric spline spaces over more general two-patch domains than trilinear ones. We will follow the idea presented in [29, 36] for the case of bivariate $C^1$–smooth multi-patch spline spaces.

**Example 6.2.** Consider the tricubic two-patch parameterization $\widetilde{F} = \left( \widetilde{F}(1), \widetilde{F}(2) \right)$, i.e. $\widetilde{F} \in \Pi^{(3,3,3)} \times \Pi^{(3,3,3)}$, shown in Figure 6.3 (left), and the $C^1$–smooth space $V_F$ from Example 6.1 for the case $p = 4$ and $k = 3$ with respect to the given trilinear two-patch geometry $F$. We construct a two-patch geometry $\widehat{F} = \left( \widehat{F}^{(1)}, \widehat{F}^{(2)} \right) \in V_F^3$ which approximates $\widetilde{F}$ “as good as possible”. This is done by performing $L^2$ approximation for each coordinate function of $\widetilde{F}$ using the basis functions of the space $V_F$. The resulting two-patch geometry $\widehat{F}$, see Figure 6.3 (right), possesses an average $L^2$-error

$$\epsilon = \frac{1}{11^3} \sum_{i \in \{1,2\}} \sum_{\ell_1=0}^{10} \sum_{\ell_2=0}^{10} \sum_{\ell_3=0}^{10} \left( \widehat{F}^{(i)}(\frac{\ell_1}{10}, \frac{\ell_2}{10}, \frac{\ell_3}{10}) - \widetilde{F}(i)(\frac{\ell_1}{10}, \frac{\ell_2}{10}, \frac{\ell_3}{10}) \right) \left( \frac{\ell_1}{10}, \frac{\ell_2}{10}, \frac{\ell_3}{10} \right)$$

less than $10^{-7}$ with respect to a bounding box with a volume of around $\frac{8}{3}$. 

Figure 6.3: The tricubic two-patch geometry $\widetilde{F}$ (left) and its $L^2$ approximation $\widehat{F}$ (right). Both parameterizations look quite identically, see. Example 6.2.
Since $\tilde{F} \in \mathcal{V}^3$, the space $\mathcal{V}_{\tilde{F}}$ of $C^1$–smooth isogeometric functions (over the two-patch geometry $\tilde{F}$) can be directly obtained from the (reference) $C^1$–smooth space $\mathcal{V}_F$ by

$$
\mathcal{V}_{\tilde{F}} = (\mathcal{V}_F \circ F) \circ \tilde{F}^{-1} = \mathcal{G}_{D_F} \circ \tilde{F}^{-1}.
$$
We analyzed the spaces of trivariate \( C^1 \)-smooth geometrically continuous isogeometric functions on volumetric two-patch domains. We introduced the notation of gluing data and used it to define glued spline functions on two-patch domains. We were able to characterize \( C^1 \)-smooth geometrically continuous isogeometric functions as push-forwards of glued spline functions.

In addition to these theoretical results, we studied several classes of gluing data in more detail. More precisely, we identified several interesting types of trilinear geometric gluing data, and we considered the generic case of degree \( Q_3 \) as well as lower degree cases.

We analyzed the generic dimension of the glued spline space for these classes of gluing data and showed how to construct locally supported basis functions. These functions, however, do not exist for all combinations of gluing data and degrees. Even if they exist, they may all take zero values on the interface, thereby indicating \( C^1 \) locking, cf. [15].
Conclusion and future work

The class of trilinear geometric gluing data is particularly promising for applications in isogeometric analysis since it leads to locally supported basis functions without locking for moderate degrees of the spline functions.

We investigated the approximation properties of these functions. In addition we also showed how to efficiently compute the basis functions using standard arithmetic (i.e., floating point numbers). We observed that the existence of locally supported interface basis functions for spline degree $p = 3$ does not suffice to provide optimal approximation power, even though these functions take non-zero values along the interface. In addition to the experimental results we also derived a theoretical justification for this surprising fact. We also confirmed that these effects are no longer present for higher polynomial degrees.

We also developed the theoretical framework to fully analyze the space of $C^1$–smooth isogeometric spline functions on trilinearly parameterized volumetric two-patch domains. On the one hand, the dimension of the space was computed, where the obtained theoretical results verify the experimentally ones from before. On the other hand, a simple basis construction was presented, which works uniformly for any degree $p \geq 3$. In addition, the single basis functions are locally supported and possess a simple explicit representation. We see this basis as an important first step for the extension of our approach to the case of trilinearly parameterized multi-patch domains by following e.g. a similar concept as in [29, 31] for the case of bivariate spline spaces.

Moreover, the extension of our approach to more general two-patch or even multi-patch domains, comparable to the class of analysis-suitable $G^1$ parameterizations (cf. [15]) in 2D, is of interest, too. The construction of a first simple example of a
two-patch domain with curved boundary and interface surfaces shows the potential of such an extension. Also, the use of degree raising at the common interface or the development of an adaptive framework could be two possible strategies to overcome the problem of the reduced approximation power in case of spline degree $p = 3$. Finally it would be of interest to explore the potential of the $C^1$–smooth isogeometric spline functions for applications in IGA, hence solving fourth order PDEs, such as the biharmonic equation.
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Parameterizations of two volumetric subdomains ( \Omega^{(1)} ) and ( \Omega^{(2)} ) with the same parameter domain, joined at the interface ( \Gamma ).</td>
<td>16</td>
</tr>
<tr>
<td>2.2</td>
<td>The two-patch domain ( \Omega ) with the two trilinearly parameterized subdomains ( \Omega^{(1)} ), ( \Omega^{(2)} ), which are joined at the common interface ( \Gamma ) and their defining vertices.</td>
<td>27</td>
</tr>
<tr>
<td>2.3</td>
<td>The relations between the algebraic varieties ( \Sigma_{\text{type}} ), which are defined by different types of gluing data ( D ).</td>
<td>30</td>
</tr>
<tr>
<td>3.1</td>
<td>Coefficients used by the interface functions (red bullets) and coefficients contributing to standard functions (blue bullets). The figure shows the control cages of both patches.</td>
<td>35</td>
</tr>
<tr>
<td>4.1</td>
<td>Numbering of the three layers of coefficients of interface basis functions for ( p = 3 ) and ( k = 3 ). From left to right: coefficients left of, on and right of the interface. Green points refer to coefficients that are discarded due to boundary conditions.</td>
<td>44</td>
</tr>
<tr>
<td>4.2</td>
<td>Function values (top, visualized by isosurfaces (left) and a level curves in a plane (right)), coefficient pattern (bottom left) and reflection lines of a particular level set (bottom middle and right) of the single type of basis function for trilinear geometric gluing data, ( p = 3 ).</td>
<td>47</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.3</td>
<td>Coefficient patterns of basis functions for trilinear geometric gluing data, $p = 4$.</td>
<td>48</td>
</tr>
<tr>
<td>4.4</td>
<td>Isosurfaces (left column) and level curves in a plane that intersects the interface transversally (right column) visualizing the values of the six different types of basis functions for trilinear geometric gluing data, $p = 4$.</td>
<td>50</td>
</tr>
<tr>
<td>4.5</td>
<td>Coefficient patterns of basis function obtained for linear gluing data, $p = 3, 4$.</td>
<td>51</td>
</tr>
<tr>
<td>5.1</td>
<td>Two-patch geometry used for $L^2$-fitting.</td>
<td>55</td>
</tr>
<tr>
<td>5.2</td>
<td>$L^2$-approximation errors for trilinear geometric gluing data of degree $p = 3$.</td>
<td>56</td>
</tr>
<tr>
<td>5.3</td>
<td>$L^2$-approximation errors for trilinear geometric gluing data of degree $p = 4$.</td>
<td>58</td>
</tr>
<tr>
<td>6.1</td>
<td>The generic two-patch domain $\Omega$ considered in Example 6.1.</td>
<td>78</td>
</tr>
<tr>
<td>6.2</td>
<td>Examples of basis functions for $p = 4, k = 3$. Top row: Function values visualized by isosurfaces (left: $\phi^{\Gamma,0}<em>{5,5}$, right: $\phi^{\Gamma,1}</em>{2,2}$). Bottom row: Function values visualized by level curves in a plane (left: $\phi^{\Gamma,0}<em>{5,5}$, right: $\phi^{\Gamma,1}</em>{2,2}$).</td>
<td>79</td>
</tr>
<tr>
<td>6.3</td>
<td>The tricubic two-patch geometry $\tilde{F}$ (left) and its $L^2$ approximation $\tilde{F}$ (right). Both parameterizations look quite identically, see. Example 6.2.</td>
<td>80</td>
</tr>
</tbody>
</table>
## List of Tables

3.1 Dimensions $\delta^S + \delta^\Gamma_{\text{cub}}$ of the space of $C^1$-smooth isogeometric splines for cubic gluing data. ............................................. 36

3.2 Dimensions $\delta^S + \delta^\Gamma_{\text{tri}}$ of the space of $C^1$-smooth isogeometric splines for trilinear geometric gluing data. ................................. 37

3.3 Dimensions $\delta^S + \delta^\Gamma_{\text{sym}}$ of the space of $C^1$-smooth isogeometric splines for trilinear gluing data corresponding to two symmetric trilinear subdomains. .................................................. 38

3.4 Dimensions $\delta^S + \delta^\Gamma_{\text{uni}}$ of the space of $C^1$-smooth isogeometric splines for gluing data $\mathcal{T}_{\text{uni}}$. ......................................................... 39

3.5 Dimensions $\delta^S + \delta^\Gamma_{\text{quad}}$ of the space of $C^1$-smooth isogeometric splines for gluing data of degree $Q_2$. ......................................................... 39

3.6 Dimensions $\delta^S + \delta^\Gamma_{\text{lin}}$ of the space of $C^1$-smooth isogeometric splines for gluing data of degree $Q_1$. ......................................................... 40

4.1 This table shows the different types of bases occurring for the specific types of gluing data. $\times$ indicates that $\dim \mathcal{V}^F_{\mathcal{F},0} = 0$ and $(\times)$ that only basis functions with zero support on the shared face were obtained, hence leading to locking. Entries with $\checkmark$ refer to local bases with desired properties at the interface. $(\checkmark)$ signals that a basis was constructed not fulfilling all requirements for local support. ........................ 45
5.1 Comparison of time needed, when using floating point and rational computations to construct the basis. . . . . . . . . . . . . . . . . . . 54

5.2 Comparison of memory needed, when using floating point and rational computations to construct the basis. . . . . . . . . . . . . . . . . . . 54
Bibliography


