$C^1$-Smooth Geometrically Continuous Test Functions on Bilinearly Parameterized Multi-Patch Domains

MASTERARBEIT
zur Erlangung des akademischen Grades
Diplom-Ingenieurin
im Masterstudium
Mathematik in den Naturwissenschaften

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Linz, April 2015
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Acknowledgements

Most of all, I would like express my thanks to my supervisor, Univ.-Prof. Dr. Bert Jüttler. I am grateful for the constant support, all the time you spent on helping me with the difficulties and especially for giving me the opportunity to work on this topic.

I would like to thank DI Dr. Mario Kapl. Without your knowledge on this subject and your technical guidance it would not have been possible to obtain these results. It has been a great pleasure to work with you.

I am thankful to Thomas Märzinger, with whom I spent so much time during the last few years. Together we made it through all the good and bad times during our studies, I am lucky to have you as my colleague and friend.

Finally I would like to express my deepest gratitude to my family. Thank you for always believing in me, even when I did not. Special thanks to my brother, who made me think about studying Technical Mathematics in the first place. I would have never achieved this without you.
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Zusammenfassung

Abstract

When considering $C^1$-smooth geometrically continuous isogeometric functions, it can be necessary to find a basis of the null space. Therefore an algorithm is presented, whose result is a matrix, which has a structure that is as banded as possible. The columns of this matrix form the basis of the coefficient space of $C^1$-smooth geometrically continuous isogeometric functions. The algorithm can be applied to bilinearly parameterized multi-patch domains, which consist of Bézier patches of arbitrary degree $(p,p)$. When applying the algorithm to two-patch domains, with patches of degree $(2,2)$, we see, that only globally supported basis functions are obtained. Besides, we obtain the same number of basis functions, independently of the level of subdivision of the patches. Hence we conclude that Bézier patches of higher degree are of interest. We then investigate two-patch geometries, with Bézier patches of degree $(3,3)$ and degree $(4,4)$. We find, that this yields locally supported basis functions. Finally we also study domains, which consist of more than two patches. Most basis functions have a similar structure to the two-patch case. Only those, which have support on the control points near to the common vertex, have to be considered explicitly. This motivates us to call them vertex basis functions. These functions are then calculated for several multi-patch domains.
1 Introduction

Isogeometric Analysis (IgA), which was introduced in 2005 in [2], has been a widespread field of research throughout the last few years. IgA is inspired by Computer Aided Design (CAD), where curves and surfaces are parametrically represented. There are various approaches to this parametric parameterization, conventionally non-uniform rational B-spline functions (NURBS) [7] are used.

Within the framework of IgA, one typically considers isogeometric functions, see [9]. These functions are obtained by composing NURBS with the inverse of NURBS geometry mappings, where both are defined on the same parameter domain.

Isogeometric functions of higher smoothness can be constructed with use of geometric continuity ($G^r$ continuity) [5], [6]. Furthermore, geometric continuity is a highly used concept when constructing multi-patch domains.

$C^r$-smooth geometrically continuous isogeometric functions on multi-patch geometries were introduced in [4]. Besides, the space of $C^r$-smooth geometrically continuous functions was considered and a theoretical concept, to construct a basis for this space, was presented.

The remainder of this thesis is structured as follows. In Section 2 we deal with the theoretical preliminaries, which are needed afterwards. In the first part we consider curves and surfaces, before turning our attention to $G^1$ continuity. Lastly we take a closer look on the space of $C^1$-smooth geometrically continuous isogeometric functions. Section 3 focusses on finding a basis for the space of $G^1$ continuous isogeometric functions. We introduce an algorithm, which calculates the basis functions for arbitrary bilinearly parameterized multi-patch domains. In the last part of this section, we apply this algorithm to different two-patch geometries. We then use the algorithm to calculate the basis functions of domains,
which consist of more than two patches in Section 4. Thereafter we consider general multi-patch domains in Section 5, where we explain how obtain a basis that consists of basis functions of first kind, edge basis functions and vertex basis functions. In Section 6 we complete the thesis with some final remarks.
2 Preliminaries

2.1 Curves and Surfaces

The Bézier representation is a well-known tool to model curves and surfaces. Although Bézier curves and surfaces have several useful properties, B-splines are often used instead. The major advantage of this generalization is that B-spline basis functions have a local support. In further chapters B-spline surfaces will be used to model the geometry of interest. For further details see [7] and [1].

2.1.1 Bézier representation

Bézier curves

A general \(n\)th-degree Bézier curve is defined as

\[
C(u) = \sum_{i=0}^{n} B_{i,n} P_i. \tag{2.1}
\]

The \(P_i = (x_i, y_i, z_i)\) are called control points, and they form the control polygon. The set \(\{B_{i,n}(u)\}\) are the Bernstein polynomials of degree \(n\), defined by

\[
B_{i,n}(u) = \binom{n}{i} u^i (1-u)^{n-i}. \tag{2.2}
\]

Bézier curves have various useful properties, which are derived from properties that hold for the basis functions \(B_{i,n}(u)\). Some of them are:

1. the convex hull property holds;
2. the variation diminishing property holds. This means that intersecting a Bézier curve with any straight line, yields in at most as many intersection points with the Bézier curve, as there are intersection points of the control polygon and the same straight line;

3. the Bézier curve is invariant under transformations, and transforming the curve is done by applying the transformation to the control points;

4. $C(0) = P_0, C(1) = P_n$, the endpoints of the curve interpolate the control points $P_0$ and $P_n$.

**Bézier surfaces**

Now that we considered Bézier curves, we want to turn our attention to surfaces, in particular tensor product surfaces, since this is one of the simplest methods for representing surfaces.

The general form of such a tensor product surface is as follows

$$S(u, v) = (x(u, v), y(u, v), z(u, v)) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{i,j} f_i(u) g_j(v), \quad (2.3)$$

where $c_1 \leq u \leq d_1$ and $c_2 \leq v \leq d_2$, e.g. the $(u, v)$ domain has a rectangular form.

We consider univariate basis functions $f_i(u), g_j(v)$. The geometric coefficients $b_{i,j}$ form a bidirectional $n \times m$ net.

With use of univariate Bernstein polynomials $B_{i,n}(u)$ and $B_{j,m}(v)$ as basis functions and a net of control points $\{P_{i,j}\}$ in equation (2.3), we get

$$S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{i,j} B_{i,n}(u) B_{j,m}(v) \quad 0 \leq u, v \leq 1, \quad (2.4)$$

the Bézier surface of degree $(n, m)$. The elements of $\{P_{i,j}\}$ are referred to as Bézier points and the whole set is called Bézier net.
Bézier surfaces have similar characteristics as Bézier curves,

1. the convex hull property holds;
2. the surface is invariant under transformations;
3. Bézier surfaces interpolate the control points of the four corner points.

The variation diminishing property, which holds for Bézier curves, does no longer exist.

2.1.2 B-splines

As we have seen, Bézier curves and surfaces have various useful properties. Nevertheless this concept also has some major disadvantages, i.e. they are defined globally, meaning that varying one point has an effect on the whole Bézier curve or surface. Therefore we will hereinafter use basis-splines (B-splines).

B-spline curves

The idea is to join together piecewise polynomial curves resulting in a B-spline curve. The Bernstein polynomials are generalized to B-spline basis functions of order $p + 1$ ($p$-degree), which are given by

$$N_{i,0} = \begin{cases} 1 & \text{if } u_i \leq u < u_{i+1} \\ 0 & \text{otherwise}, \end{cases} \quad (2.5)$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,0} + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \quad p > 0, \quad (2.6)$$

with a monotonically increasing sequence $U = \{u_0, ..., u_m\}$ of real numbers, which have to satisfy $u_i \leq u_{i+1}$ and $u_i < u_{i+p+1}$. Every $u_i$ is called a knot and $U$ is referred to as the knot vector.
For a non-uniform knot vector

\[ U = (a, \ldots, a, u_{p+1}, \ldots, u_{m-p-1}, b, \ldots, b), \]

a \textit{pth-degree B-spline curve} is defined as follows

\[ C_p(u) = \sum_{i=0}^{n} N_{i,p}(u) P_i \quad a \leq u \leq b, \quad (2.7) \]

where \( \{P_i\} \) denote again the \textit{control points} and \( \{N_{i,p}(u)\} \) are the B-spline basis functions of degree \( p \). An internal knot might appear multiple times, but \( a < u_i < u_{i+p} < b \), always has to be fulfilled. This means that a unique knot from the interior of the knot span may not appear more than \( p \) times. The \textit{control polygon} is again determined by the control points. In the following we always assume that \( a = 0 \) and \( b = 1 \).

Besides being defined locally, B-spline curves have similar geometric properties to the Bézier curves:

1. B-spline curves are a generalization of Bézier curves, if we choose \( n = p \) and \( U = \{0, \ldots, 0, 1, \ldots, 1\} \), the curve \( C(u) \) it a Bézier curve;
2. if we have a \( p \)th-degree B-spline curve \( C(u) \), with \( n + 1 \) control points and \( m + 1 \) knots, those values are related by \( m = n + p + 1 \);
3. the curve interpolates the control points at the endpoints, i.e. \( P_0 = C(0), P_n = C(1) \);
4. B-spline curves are invariant under affine transformations. The transformation is applied to the control points in order to transform the curve;
5. the convex hull property holds;
6. the variation diminishing property holds.

In further chapters, we will use derivatives of B-splines, therefore we will first define derivatives of the B-spline basis functions. The first derivative of a B-spline basis function

\[ N_{i,p}(u) = \text{...} \]
is defined by

\[ N'_{i,p} = \frac{p}{u_{i,p} - u_i} N_{i,p-1}(u) - \frac{p}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u). \] (2.8)

By repeatedly applying equation (2.8) we get the \( k \)th derivative

\[ N^{(k)}_{i,p}(u) = \frac{p}{u_{i+p} - u_i} N^{(k-1)}_{i,p-1}(u) - \frac{p}{u_{i+p+1} - u_{i+1}} N^{(k-1)}_{i+1,p-1}(u). \] (2.9)

If we have the \( k \)th derivative of the basis function for a fixed value \( u \), the \( k \)th derivative of a B-spline curve \( C(u) \) is defined by

\[ C^{(k)}(u) = \sum_{i=0}^{n} N^{(k)}_{i,p}(u) P_i. \] (2.10)

It is also possible to calculate derivatives of the B-spline curve, by differentiating the control points. This is done recursively, and yields in

\[ C^{(k)}(u) = (p - 1)(p - 2) \cdots (p - k) \sum_{i=0}^{n-k} N_{i,p-k}(u) P_i^{(k)}. \] (2.11)

with

\[ P_i^{(k)} = \begin{cases} P_i & k = 0 \\ \frac{P_i^{(k-1)} - P_{i-1}^{(k-1)}}{u_{i+k-j} - u_i} & k > 0. \end{cases} \] (2.12)

The knot vector for the \( k \)th derivative is then given by

\[ U^{(k)} = (0, \ldots, 0, u_{p+1}, \ldots, u_{m-p-1}, 1, \ldots, 1). \]

On the inner of a knot vector all derivatives of the basis functions \( N_{i,p}(u) \) exist. If a knot has multiplicity \( k \) then \( N_{i,p}(u) \) is only \( p - k \) times continuously differentiable at this knot. Since \( C(u) \) is a linear combination of the basis functions \( N_{i,p}(u) \), the same holds for B-spline curves.
**B-spline surfaces**

Analogously to Bézier surfaces we can now define *B-spline surfaces* of $p$th-degree in the $u$ direction and $q$th-degree in the $v$ direction

$$S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}(u) N_{j,q}(v) P_{i,j}, \quad (2.13)$$

with a product of univariate B-spline basis functions $N_{i,p}(u)$, $N_{j,q}(v)$ and a net of control points $\{P_{i,j}\}$. For a B-spline surface we have two knot vectors $U$ and $V$ with $r + 1$ and $s + 1$ knots respectively

$$U = (0, \ldots, 0, u_{p+1}, \ldots, u_{r-p-1}, 1, \ldots, 1),$$

$$V = (0, \ldots, 0, v_{q+1}, \ldots, v_{s-q-1}, 1, \ldots, 1).$$

B-spline surfaces have similar characteristics to the Bézier surfaces.

1. B-spline surfaces are a generalization of Bézier surfaces;

2. the control points in the four corner of the surface interpolate the surface;

3. B-spline surfaces are invariant under affine transformations;

4. the convex hull property holds;

5. additionally B-spline surfaces are defined locally;

6. differentiability and continuity are derived from the corresponding properties of the basis functions. Therefore the surface $S(u, v)$ is, at a $u$ knot with multiplicity $j$, $p - j$ times differentiable in the $u$ direction. Similarly the surface is in $v$ direction $q - j$ times differentiable at a $v$ knot with multiplicity $j$.

Note that the variation diminishing property does not hold for B-spline surfaces.
Just as before, derivatives of \( S(u, v) \), for fixed values \((u, v)\), are given with use of the derivatives of the basis functions, equation (2.8), by

\[
\frac{\partial^{k+l}}{\partial^k u \partial^l v} S(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_i^{(k)}(u) N_j^{(l)}(v) P_{i,j}. \tag{2.14}
\]

A formal differentiation of a B-spline surface results in

\[
\frac{\partial^{k+l}}{\partial^k u \partial^l v} S(u, v) = \sum_{i=0}^{n-k} \sum_{j=0}^{m-l} N_{i,p-k}(u) N_{j,q-l}(v) P_{i,j}^{(k,l)}, \tag{2.15}
\]

with

\[
P_{i,j}^{(k,l)} = (q - l + 1) \frac{P_{i,j+1}^{(k,l-1)} - P_{i,j}^{(k,l-1)}}{v_{j+q+1} - v_{j+l}}. \tag{2.16}
\]

### 2.1.3 Knot insertion and degree elevation

A B-spline curve is approximated by the control polygon, which is a piecewise linear curve. To obtain more degrees of freedom and to improve the approximation, two techniques are introduced, knot insertion and degree elevation, for further details see [7], [8] and [2].

#### Knot insertion

Suppose we have a B-spline curve \( C(u) = \sum_{i=0}^{n} N_{i,p}(u) P_i \), with corresponding knot vector \( U = (u_0, ..., u_m) \), and we want to insert \( \bar{u} \in [u_k, u_{k+1}) \). The new knot vector is then given by \( \bar{U} = (u_0, ..., u_k, \bar{u}, u_{k+1}, ..., u_m) \). With use of equations (2.5) and (2.6) we obtain the \( n+1 \) new basis functions. The new control points can be computed, using the initial control points \( P_i \), by

\[
\bar{P}_i = \alpha_i P_i + (1 - \alpha_i) P_{i-1}, \tag{2.17}
\]

with

\[
\alpha_i = \begin{cases} 
1 & i \leq k - p, \\
\frac{\bar{u} - u_i}{u_{i+p} - u_i} & k - p < i \leq k, \\
0 & k < i.
\end{cases} \tag{2.18}
\]
It does not necessarily have to be a new knot value which is inserted, but as mentioned before, the curve becomes discontinuous, if a unique knot appears more than \( p \) times.

To insert a knot into a surface, we apply the same strategy to the control net, in particular to the rows and/or columns. Let \( S(u, v) \) be a B-spline surface with control points \( P_{i,j} \), say \( i \) is the row index \( 0 \leq i \leq n \) and \( j \) the column index \( 0 \leq j \leq m \). If we want to insert a knot \( \bar{u} \), we have to add it into each of the \( m+1 \) columns. The same applies when adding a knot \( \bar{v} \), which we have to insert into the \( n+1 \) rows of control points. This results in the new control points \( \bar{P}_{i,j} \).

The insertion of a knot does not change the curve or surface parametrically nor geometrically.

**Degree elevation**

Consider a \( p \)-th-degree B-spline curve \( C_p(u) \), as in equation (2.7), and let \( U \) be the corresponding knot vector. The same piecewise polynomial curve, in higher dimensional space, has then the form

\[
C_{p+1}(u) = \sum_{i=0}^{\hat{n}} N_{i,p+1} \bar{P}_i,
\]

with a new knot vector \( \hat{U} \).

Now suppose that the interior of \( U \) includes a knot of multiplicity two or higher, e.g.

\[
U = (u_0, \ldots, u_m) = (a, \ldots, a, u_1, \ldots, u_i, \underline{u_i}, \underline{u_i}, u_{i+m_1}, \ldots u_n, b, \ldots, b).
\]

The curve \( C_p(u) \) is \( C^{p-m_i} \)-smooth at a knot \( u_i \) with multiplicity \( m_i \). The same has to hold for \( C_{p+1}(u) \). Therefore the same knot \( u_i \) must have multiplicity \( m_i + 1 \) in the knot vector \( \hat{U} \), i.e.

\[
\hat{U} = (u_0, \ldots, u_{\hat{m}}) = (a, \ldots, a, u_1, \ldots, u_i, \underline{u_i}, \underline{u_i}, u_{i+m_{i+1}}, \ldots u_n, b, \ldots, b).
\]
From this follows $\hat{m} = m + s + 2$ and $\hat{n} = n + s + 1$.

There are different ways to compute the new control points $\hat{P}_i$. Intuitively, one can set

$$\sum_{i=0}^{n} N_{i,p}(u) P_i = \sum_{i=0}^{\hat{n}} N_{i,p+1}(u) \hat{P}_i.$$ 

To generate a system of $\hat{n} + 1$ linear equation, and compute the $\hat{P}_i$, we only have to evaluate at appropriate $\hat{n} + 1$ points. Unfortunately this is an inefficient method, therefore another concept is used.

This method consists of three steps, for further details see [7]. The first step is to subdivide the curve into Bézier segments, this is done by knot insertion. Subsequently each of the Bézier curves is degree elevated. Finally to combine the pieces, all unnecessary knots, which separate adjacent segments, are removed and the result is one B-spline curve with higher degree.

Similarly to knot insertion for surfaces, we have to apply the method for degree elevation to the rows and columns of the control net to accomplish degree elevation for a surface. To elevate the degree $p$, which corresponds to the $u$ direction, we have to apply the method to all of the $m + 1$ columns of the control net. Analogously applying the method to each of the $n + 1$ rows, yields in elevation of degree $q$ in $v$ direction.

As knot insertion, degree elevation does not change the B-spline surface or curve geometrically and parametrically.

### 2.2 Geometric continuity

In further chapters, the main task will be to join together Bézier patches. One possibility is to join them together with $C^r$ continuity, meaning that at the junction the segments have to be the same up to the $r$th derivative. Another approach is to use $G^r$ continuity.
(geometric continuity), where only the shape of the curve or surface is of interest. One advantage of $G^r$ continuity is, that it is invariant under reparameterization. Furthermore geometric continuity is helpful, when considering more general topologies, since we can easily combine several Bézier patches. We restrict ourselves to $G^1$ continuity, since only this is needed in further chapters. For a generalization and more details see [1], [3], [5] and [6].

### 2.2.1 $G^1$ continuity for curves

We start with the easiest case, which is, that two curves meet in a common point. Generally $G^1$ continuity for such two $C^1$ curves, joining in one common point, means, that their tangent vectors at the joining point have the same direction after a reparameterization.

A formal definition is, that two $C^1$ curves $X(u)$ for $u \in [u_0, u_1]$ and $Y(v)$ for $v \in [v_0, v_1]$ join at a regular point $p = X(u_1) = Y(v_0)$ with $G^1$ continuity, if their first derivatives are identical at $p$ after reparameterization. To compare the derivatives of $X(u)$ and $Y(v)$, they have to be evaluated with respect to the same parameter. Therefore we can reparameterize for example $X(u)$, using $u \mapsto u(v)$ with $u(v_0^-) = u_1^-$. We use ”+” to indicate evaluation from the right, and ”-“ refers to evaluation from the left. It is important to note, that the map, which is used to reparameterize one of the curves, has to be orientation-preserving. Then

$$
\frac{d}{dv} Y(v)|_{v_0^+} = \frac{d}{dv} X(u(v))|_{u = u(v_0^-)}, \quad (2.21)
$$

means that the curves meet with $G^1$ continuity in $p$.

### 2.2.2 $G^1$ continuity for surfaces

$G^1$ continuity in a common point

We also want to extend the concept of geometric continuity, especially $G^1$ continuity, to surfaces. Again we want to consider two surface, which join with $G^1$ continuity in a regular
point \( p \), which now means that they have the same tangent plane in \( p \).

Suppose we have two surfaces \( X(u, v), Y(s, t) \in \mathbb{R}^3 \) that satisfy \( \partial_u X \times \partial_v X \neq 0 \) and \( \partial_s Y \times \partial_t Y \neq 0 \) for \( u \in [u_0, u_1], v \in [v_0, v_1], s \in [s_0, s_1] \) and \( t \in [t_0, t_1] \). Assume that they meet in a point \( p = X(\bar{u}, \bar{v}) = Y(\bar{s}, \bar{t}) \). In order to establish \( G^1 \) continuity, we have to compare their derivatives in \( p \). Therefore we need to reparameterize, as before, with an orientation-preserving transformation. One possibility is to use \( u \mapsto u(s, t) \) and \( v \mapsto v(s, t) \). \( G^1 \) continuity for those surfaces in \( p \) is then defined by

\[
\partial_s Y(s, t) = \partial_u X(u(s, t), v(s, t)) + \partial_v X(u(s, t), v(s, t)),
\]

\[
\partial_t Y(s, t) = \partial_u X(u(s, t), v(s, t)) + \partial_v X(u(s, t), v(s, t)),
\]

(2.22)

evaluated for \((s, t) = (\bar{s}, \bar{t})\).

**\( G^1 \) continuity along a common interface**

Now we want to consider the case that two surfaces do not just join in one regular point, but along a common boundary curve. Therefore let us recall the following definitions:

1. a domain \( D \) will in the following always be a simple closed subset of \( \mathbb{R}^2 \), which is supposed to be bounded by edges \( E_i, i < \infty \).

2. For a set \( T, \varepsilon(T) \) is supposed to be an open neighbourhood. If \( E_i \) is an edge of \( D^{(i)} \) and \( E_j \) an edge of \( D^{(j)} \), then \( r : \varepsilon(E_i) \to \varepsilon(E_j) \) is a \( C^1 \) reparameterization between \( D^{(i)} \) and \( D^{(j)} \). \( r \) has to be \( C^1 \) continuous and invertible and it has to map \( E_i \) to \( E_j \). Furthermore it also has to map exterior points of \( D^{(i)} \) to interior points of \( D^{(j)} \).

3. A map \( g : \mathcal{D} \to \mathbb{R}^d \) is a \( C^1 \) geometry map, if the first derivative \( Dg \) is continuous and \( \det(Dg) \) does not vanish. Then \( g(\mathcal{D}) \) is referred to as a \( C^1 \) patch.

Let the two surfaces \( X, Y \) now be \( C^1 \) geometry maps. They then meet along a common boundary curve \( Y(E) \), with \( G^1 \) continuity, via the \( C^1 \) reparameterization \( r \), if they satisfy

\[
\partial Y|_E = \partial X \circ r|_E.
\]

(2.23)
This means, that the two surfaces have to fulfil the requirements for $G^1$ continuity at every point along the common boundary curve $E$.

For the sake of simplicity, we now assume $0 \leq u, v, s, t \leq 1$ for the two surfaces $X(u, v)$ and $Y(s, t)$. If we use the regular reparameterization $v \mapsto v(t)$ on $[0, 1]$, the common boundary curve is given by

$$X(1, v(t)) = Y(0, t).$$

(2.24)

Since we want to obtain $G^1$ continuity, the tangent planes of the two surfaces have to be identical along the common interface. Hence the following equation has to be fulfilled

$$a(t) \partial_s Y(0, t) + b(v(t)) \partial_u X(1, v(t)) + c(v(t)) \partial_v X(1, v(t)) = 0,$$

(2.25)

with $a(t) \neq 0, b(v(t)) \neq 0$. This is equivalent to the determinant form

$$\det(\partial_s Y(0, t), \partial_u X(1, v(t)), \partial_v X(1, v(t))) = 0.$$  

(2.26)

A typical choice would be $v(t) = t$.

### 2.2.3 N-vertex problem

So far we have defined $G^1$ continuity for two surfaces, if they meet in one point and along common edges. Now we also want to give the definition for a common corner $Q \in \mathbb{R}^3$ of more than two surfaces. This point $Q$ can either be enclosed by the geometry maps, or be a point on the global boundary, e.g. the segments meet without enclosing the point.

Suppose that we have the following $C^1$ geometry maps $g^{(i)} : D^{(i)} \to \mathbb{R}^3$ for $i = 1, \ldots, n$. They meet with $G^1$ continuity via the reparameterization $r_{j,j+1}, j = 1, \ldots, n-1$, with corner $Q \in \mathbb{R}^3$, if

- $g^{(i)}$ and $g^{(i+1)}$, two adjacent segments, meet with $G^1$ continuity via $r_{i,i+1}$ along their common edge $g^{(i+1)}(E_{i+1})$. 

• the corner $Q$ fulfils $Q = g^{(i)}(E_i(0))$,

• the tangent sectors, defined by the normalized tangent vectors of every $g^{(i)}$, lie in the common tangent plane and they can touch each other but they may not overlap.

This ensures that the corner is not encircled by the geometry maps more than once.

These conditions ensure that several segments meet at a common vertex on the global boundary. For a vertex $Q$, to be fully enclosed by geometry maps, we additionally need that

• the $C^1$ geometry maps join in a cycle and therefore $g^{(n)}$ and $g^{(1)}$ have to join with $G^1$ continuity via $r_{n,1}$ along their common edge $g^{(1)}(E_1)$.

In the case that a vertex $Q$ is enclosed by more than two segments, problems can occur and the "vertex enclosure problem" is possibly not solvable. Hence additional conditions are needed. One possibility are twist compatibility constraint. Hereinafter no further details are needed, since we fulfil the constraints implicitly, but they are described in [1] and [5].

### 2.3 The space of $C^1$-smooth geometrically continuous isogeometric functions

In the following we will consider $G^1$ continuity for B-spline surfaces, which are tensor product surfaces. The tensor product B-spline space of degree $p_i \in \mathbb{N}_0^2$ is denoted as $S^{(i)}$. Furthermore we want to consider $[0, 1]^2$ as our initial domain $D$. Then we can define $n$ initial geometry mappings, which are now supposed to be bilinear Bézier patches, by

$$g^{(i)} : [0, 1]^2 \to \mathbb{R}^2,$$

and

$$\xi^{(i)} = (\xi_1^{(i)}, \xi_2^{(i)}) \mapsto (g_1^{(i)}, g_2^{(i)}) = g^{(i)}(\xi^{(i)}) \quad i \in \{1, ..., n\}.$$
It follows $g^{(i)} \in S^{(i)}$.

As stated before, each geometry mapping $g^{(i)}$ defines a $C^1$ patch,

$$\Omega^{(i)} = g^{(i)}([0, 1]^2).$$

The union of all $C^1$ patches $\Omega^{(i)}$ is referred to as the computational domain $\Omega \subset \mathbb{R}^2$. Besides we suppose, that

$$\Omega^{(i)} \cap \Omega^{(k)} = \emptyset \quad i, k \in 1, ..., n; i \neq k.$$ 

Since we consider Bézier patches, each geometry mapping can be expressed in terms of B-spline basis functions $N_{j,p}^{(i)} : [0, 1] \to \mathbb{R}^2$ and control points $P_j^{(i)} \in \mathbb{R}^2$. This yields in

$$g^{(i)}(\xi^{(i)}) = \sum_{j \in J_i} P_j^{(i)} N_{j,p}^{(i)}(\xi^{(i)}). \quad (2.27)$$

The set $J_i$ is supposed to be an appropriate index set.

We are interested in isogeometric functions, which are given as the composition of a B-spline patch and the inverse of the geometry mapping. Hence for every $\Omega^{(i)}, i \in \{1, ..., n\}$, the space of globally $C^1$-smooth isogeometric functions is

$$V^{(i)} = \left\{ v \in C^1(\Omega^{(i)}) : v \in S^{(i)} \circ (g^{(i)})^{-1} \right\}. \quad (2.28)$$

In the following we do not just want to consider isogeometric functions on one $C^1$ patch, but on the whole computational domain. Therefore we define the space of globally $C^1$-smooth isogeometric functions on $\Omega$ by

$$V = \left\{ v \in C^1(\Omega) : v|_{\Omega^{(i)}} \in S^{(i)} \circ (g^{(i)})^{-1} \forall i \in \{1, ..., n\} \right\}. \quad (2.29)$$

As mentioned before, we want to consider geometric continuity instead of parametric
continuity, on this account we introduce graph surfaces. Suppose we have two adjacent patches \( \Omega^{(i)} \) and \( \Omega^{(k)} \), which join along a common boundary curve \( E^{(ik)} = \Omega^{(i)} \cap \Omega^{(k)} \). Each isogeometric function \( w^{(i)} \in V^{(i)} \) has an associated graph surface, denoted as \( F^{(i)} \), which is defined by

\[
F^{(i)}(\xi^{(i)}) = (g^{(i)}(\xi^{(i)}), \omega^{(i)}(\xi^{(i)}))^T,
\]

where \( \omega^{(i)} \in S^{(i)} \). And similarly each element \( w^{(k)} \in V^{(k)} \) has an associated graph surface \( F^{(k)} \). Those graph surfaces then meet with \( G^1 \) continuity along the common boundary curve.

This leads us to the following theorem, see [3].

**Theorem 1.** An isogeometric function \( w : \Omega \rightarrow \mathbb{R} \) is contained in the space \( V \), given in (2.29), if and only if the graph surfaces \( F^{(i)} \) and \( F^{(k)} \), for all adjacent and disjoint patches \( \Omega^{(i)} \) and \( \Omega^{(k)} \), \( i, k \in \{1, ..., n\} \), join at their common boundary curve with \( G^1 \) continuity.

Thus the space \( V \) will now be referred to as the space of \( C^1 \)-smooth geometrically continuous isogeometric functions.
3 Basis for $G^1$ continuous isogeometric functions

So far geometrically continuous isogeometric functions were introduced. To use them further on, it is necessary to find a basis of the space $V$. A theoretical approach for NURBS basis functions and general $C^s$-smooth geometrically continuous isogeometric functions was introduced in [3]. This idea will now be adapted to the special setting used here. Then a general algorithm to find basis functions is presented.

3.1 Construction of a basis for $G^1$ continuous isogeometric functions

From equation (2.28), it follows that each element $w^{(i)} \in V^{(i)}$ has a representation via an element $\omega^{(i)} \in S^{(i)}$

$$w^{(i)}(x) = (\omega^{(i)} \circ (g^{(i)})^{-1})(x), \quad x \in \Omega^{(i)}.$$  (3.1)

Two adjacent patches $\Omega^{(i)}$ and $\Omega^{(k)}$ join with $C^1$ continuity along their common boundary curve $E^{(i,k)}$, if their first derivatives of the isogeometric functions $w^{(i)}$ and $w^{(k)}$ are identical along $E^{(i,k)}$. Hence

$$(\partial_1^j \partial_2^k w^{(i)})(x) = (\partial_1^j \partial_2^k w^{(k)})(x), \quad x \in E^{(i,k)},$$  (3.2)
with $j + \ell \leq 1$ has to be fulfilled. As $x = (x_1, x_2)$ we denote the global coordinates with regard to $\Omega$.

Moreover, since we consider tensor-product patches, each $\omega^{(i)} \in S^{(i)}$ can be represented with use of basis functions $(N_{j,p}^{(i)})_{j \in J_i}$. This yields

$$\omega^{(i)}(\xi^{(i)}) = \sum_{j \in J_i} b_j^{(i)} N_{j,p}^{(i)}(\xi^{(i)}). \quad (3.3)$$

We can conclude, that equation (3.2) can now be written as

$$\sum_{j \in J_i} b_j^{(i)}(D_1 D_2(N_{j,p}^{(i)} \circ (g^{(i)})^{-1}))(x) = \sum_{j \in J_k} b_j^{(k)}(D_1 D_2(N_{j,p}^{(k)} \circ (g^{(k)})^{-1}))(x) \quad x \in E^{(i,k)}. \quad (3.4)$$

The space $S^{(i)}$ is of finite dimension, therefore we obtain finitely many linear restrictions for the coefficients $b_j^{(i)}$ and $b_j^{(k)}$.

Let us take a closer look at bilinearly parametrized two-patch domains. We consider a geometry $\Omega = \Omega^{(1)} \cup \Omega^{(2)}$, that consists of two adjacent Bézier patches $\Omega^{(1)}$ and $\Omega^{(2)}$. As described above, these patches are supposed to be defined by two bilinear geometry mappings $g^{(1)}$, $g^{(2)}$. Analogously to equation (2.24), we can construct a common boundary curve, given as

$$g^{(1)}(1, \xi) = g^{(2)}(0, \xi),$$

with $\xi = \xi_2^{(1)} = \xi_2^{(2)} \in [0, 1]$. Hence, it follows for $w \in V$ and $\omega^{(k)} \in S$

$$w \text{ is continuous along the common interface } \iff \omega^{(1)}(1, \xi) = \omega^{(2)}(0, \xi).$$

From Theorem 1, we already know, that we have to take the associated graph surfaces $F^{(1)}$ and $F^{(2)}$ into account, in order to obtain geometric continuity along the common edge. Our goal is to join together these two patches with $G^1$ continuity along their common
interface, hence
\[ \det((\partial_1 F^{(1)}(1, \xi), (\partial_1 F^{(2)})(0, \xi), (\partial_2 F^{(2)})(0, \xi))) = 0, \quad \xi \in [0, 1] \]  
(3.5)

has to be fulfilled, see equation (2.26).

This yields a homogeneous system for the coefficients \( b^{(i)}_j \) and \( b^{(k)}_j \) in equation (3.4), which can be represented by
\[ S \mathbf{b} = 0, \quad \mathbf{b} = (b^{(i)}_j)_{j \in J, i \in \{1, \ldots, n}\}. \]  
(3.6)

We know, that equation (3.5) can be written as
\[ a(\xi) \partial_1 \omega^{(2)}(0, \xi) + b(\xi) \partial_2 \omega^{(1)}(1, \xi) + c(\xi) \partial_1 \omega^{(1)}(1, \xi). \]  
(3.7)

With a comparison of the coefficient functions
\[
\begin{align*}
    a(\xi) &= \partial_2 g^{(1)}_1(1, \xi) \partial_1 g^{(1)}_2(1, \xi) - \partial_1 g^{(1)}_1(1, \xi) \partial_2 g^{(1)}_2(1, \xi), \\
    b(\xi) &= \partial_1 g^{(2)}_1(0, \xi) \partial_1 g^{(1)}_2(1, \xi) - \partial_1 g^{(1)}_1(1, \xi) \partial_1 g^{(2)}_2(0, \xi), \\
    c(\xi) &= \partial_2 g^{(1)}_1(1, \xi) \partial_1 g^{(2)}_2(0, \xi) - \partial_1 g^{(2)}_1(0, \xi) \partial_2 g^{(1)}_2(1, \xi),
\end{align*}
\]

we can solve the linear system in equation (3.6).

In the following we will present an algorithm, that generates a basis of the space \( V \), where every basis function fulfils the condition in equation (3.6).

### 3.2 Algorithm for the construction of basis functions

First, we want to consider a computational domain \( \Omega \), which consists of two adjacent but disjoint four-sided patches \( \Omega^{(1)} \) and \( \Omega^{(2)} \). They are supposed to be a bilinear representation of the plane and might therefore be accordingly degree-elevated.
We want to have B-spline representations of the geometry mapping $g^{(1)}$ and $g^{(2)}$, which are biquartic Bézier patches, as described before. In order to obtain this, we uniformly insert knots in both parameter directions. We will see that it is sufficient to consider B-spline representations of degree (3,3) and of degree (4,4). To obtain the representation of degree (3,3), we have to insert $m \geq 0$ equidistant knots, each one with multiplicity 2. The same holds for degree (4,4), where the inner knots have to have multiplicity 3.

In the case of a two-patch domain, two different kinds of functions occur in the basis of $V$, further described in [3].

1. Basis functions with a support that is contained in one of the patches, are called *basis functions of the first kind*. Those basis functions are not affected by any control points, which lie on the common boundary curve or the neighbouring columns. Hence they can be constructed with use of equation (2.28), i.e.

$$N_{i,p}^{(i)} \circ (g^{(i)})^{-1}.$$  \hspace{1cm} (3.8)
Control points that correspond to those basis functions are displayed in black in Fig. 3.1.

2. On the other hand, those basis functions with support in both patches, are denoted as *basis functions of the second kind*. In contrast to the first kind, only those control points on the common boundary curve and the neighbouring columns contribute to the basis functions of the second kind, shown in green in Fig. 3.1. All the other control points have corresponding coefficients $b_j^{(i)}$ equal to zero.

For domains, which consist of more than two subdomains a third kind will be introduced.

If we consider equation (2.30), the graph surface, in more detail, we see that the first two coordinates $g(i)(\xi(i)) = (g_1(i)(\xi(i)), g_2(i)(\xi(i)))$ are just the ones of the corresponding initial geometry mapping. With use of Theorem 1, we can conclude, that only the third coordinate of the associated graph surface is of interest, in order to guarantee that an isogeometric function $w$ is an element of the space $V$.

Basis functions of first kind are not of further interest, since they do not affect $G^1$ continuity along the common boundary curve. Furthermore each function in $V$ can uniquely be represented with use of basis functions of first kind and basis functions of second kind. Hence the direct sum of the set of basis functions of first kind and the set of basis functions of second kind forms the basis of $V$. For further details see [4]. Because of that, we do not have to consider basis functions of first kind in the following. Our objective is to find a sparse basis, with local support, for basis functions of second kind.

We want to introduce a new algorithm, with which it is possible to construct a basis for the space of geometrically continuous isogeometric functions of second kind. It is important to note, that we still restrict ourselves to the special case of $G^1$ continuity.

Assume, that after having inserted equidistant inner knots in both patches $\Omega^{(1)}$ and $\Omega^{(2)}$, there are in total $k$ control points, which lie on the common interface and the two
neighbouring columns, the ones which are shown in green in Fig. 3.1. They have a B-spline representation, e.g.

\[
p_i = \begin{pmatrix} x_i \\ y_i \\ w_i \end{pmatrix}, \quad i = 0, ..., k - 1.
\]

In order to see a structure in the arrangement of individual control points in each basis vector, all control points, which contribute to basis functions of second kind, are sorted as shown in Fig. 3.2.

In order to get \(G^1\) continuity along the common interface, we have to choose the third coordinate of each control point \(p_i\), such that the smoothness condition in equation (3.6) is fulfilled. This can be achieved by using algorithm 3.2.

The algorithm generates a \((k \times l)\) matrix \(M\), which is as banded as possible, whenever a basis with this condition exists. The number of columns \(l \leq k\) also denotes the number of basis function of second kind. Each column refers to one basis function and consists of the coefficients of this function.

Figure 3.2: Sorting of control points for Bézier patches of degree (4,4), depending on the location of each point in the initial geometries. The algorithm, which follows, uses this configurations.
Let $W = \{w_0, \ldots, w_{k-1}\}$ refer to the set of the unknown third coordinates. Furthermore we define $\{SW = 0\} = C$ and $Z(X) = \{w_i = 0| i \in X\}$. With $Z(U)$ and $Z(L)$ we will denote upper constraints and lower constraints respectively.

Algorithm 1 Construction of banded basis

```
Input: \( k \) control points \( C = \{SW = 0\} \)
1: set \( L = \{1, \ldots, k-1\} \); \( U = \emptyset \); \( \ell = 1 \)
2: repeat
3: \( \text{while } C \cup Z(U) \cup Z(L) \text{ results in trivial solution only do} \)
4: \( \text{delete min } L \text{ in } L \)
5: \( \text{end while} \)
6: solve \( C \cup Z(U) \cup Z(L) \)
7: store the solution \( W^{(\ell)} \) in the \( \ell \)th column of \( M \)
8: \( \text{find min}_{i \in \{0, \ldots, k-1\}} w_i^{(\ell)} \neq 0 \)
9: \( \text{U} = \text{U} \cup \{i\} \)
10: \( \text{until } L = \emptyset \)
```

It was stated before, that basis functions of first kind and basis functions of second kind can be investigated separately. Basis functions of first kind can be constructed with use of equation (3.8). Basis functions of second kind are calculated with algorithm 3.2. The following theorem guarantees that the constructed matrix \( M \) consists of all coefficients of basis functions, which span the linear space of basis functions of second kind. Remember, that each column of \( M \) refers to one basis function.

Note that \( U \cup L \) refers to the set of all coefficients, which are set to zero. The set of coefficients not equal to zero is denoted as \( N = W \setminus (U \cup L) \).

**Theorem 2.** The basis functions, calculated by algorithm 3.2, span the whole linear space of functions of second kind.

**Proof.** Suppose that in the \( j \)th loop of algorithm 3.2, there is a non-trivial solution. Furthermore suppose, that \( \mathcal{N} = W \setminus (U \cup L) = \{w_i, \ldots, w_m\} \), with \( 0 < i < m < k - 1 \), is the
set of coefficients not equal to zero. This means that there exists a matrix $A$, such that

$$A\vec{w} = 0, \quad \vec{w} = \begin{pmatrix} w_i \\ \vdots \\ w_m \end{pmatrix}$$

In the next loop, we delete the minimal element in $\mathcal{L}$, hence we get $\mathcal{N}_+ = \{w_i, ..., w_{m+1}\}$, with $0 < i < m + 1 \leq k - 1$. Furthermore there exists a matrix $A_+$, such that

$$A_+\vec{w}_+ = 0, \quad \vec{w}_+ = \begin{pmatrix} w_i \\ \vdots \\ w_{m+1} \end{pmatrix}$$

holds. We define the spaces $G = \{\vec{w} \mid A\vec{w} = 0\}$ and $H = \{\vec{w}_+ \mid A_+\vec{w}_+ = 0\}$. It has to be proved, that $\dim G + 1 \geq \dim H$ holds.

For an arbitrary space $F = \{\vec{x} \in \mathbb{R}^n \mid B\vec{x} = 0\}$, the dimension is defined as $\dim F = n - \rank(B)$.

It is obvious that

$$\rank(A) \leq \rank(A_+) \iff -\rank(A) \geq -\rank(A_+) \iff m - \rank(A) + 1 \geq m + 1 - \rank(A_+)$$

holds. With use of the definition of the dimension of a space, we get

$$\dim G + 1 \geq \dim H.$$
is calculated, we find the minimal element in \( N \), and set it equal to zero, \( U = U \cup \{i\} \). Hence, any solution, which is calculated in further loops, is linearly independent.

### 3.3 Two-patch domains

Algorithm 3.2 can be used to construct basis functions of second kind for different kinds of computational domains. In the following it is applied to generic cases only, which are described in more detail in [4]. First of all we investigated a geometry, that consists of two bilinearly parameterized Bézier patches.

We investigate on the one hand for which degree \((d, d)\) the basis functions have a local support. On the other hand we are interested in the structure of the different basis function that form the basis for \( C^1 \)-smooth geometrically continuous isogeometric functions of second kind.

#### 3.3.1 Degree (2,2)

The first geometry we considere is a two-patch domain, which consists of two neighbouring segments along the common interface in each Bézier patch of degree \((2, 2)\). An example of such a setting is shown in Fig. 3.3.

It is shown, that all control points, which are of interest for basis functions of second kind, are similarly sorted to the case of degree \((4, 4)\) Bézier patches. This will be done similarly for all further two-patch geometries.

With use of algorithm 3.2, the basis function of second kind can then be calculated. The results are shown in Fig. 3.4.

Fig. 3.4 depicts the matrix \( M \), which is calculated by the algorithm. The figure to the left, displays the coefficients of the basis functions, for the case that each Bézier patch is subdivided once, such that each one consists of two subdomains along the common
Figure 3.3: A geometry, which consists of two Bézier patches of degree $(2, 2)$, which join along a common edge. Control points which are considered in the algorithm, are displayed in green. They are sorted as before.

Figure 3.4: Coefficients of basis functions of second kind for two adjacent segments in each Bézier patch of degree $(2, 2)$.

boundary curve, see Fig. 3.3. The one to the right depicts the coefficients for a geometry, where each patch consists of three segments along the common interface. For both cases, there are five columns, where each refers to one basis function of second kind. For each function, coefficients unequal to zero are displayed in blue. It can be seen, that for degree
(2, 2) Bézier patches, only globally supported basis functions are obtained. Moreover the number of obtained basis functions does not change, if there are more knot spans inserted (further subdivision) along the common edge. Therefore we will now turn our attention to higher degree domains.

### 3.3.2 Degree (3,3)

We consider two degree (3, 3) Bézier patches. Each of them can be subdivided, such that every one consists of two neighbouring pairs of segments along the common boundary curve. It is also possible to further subdivide them, such that each consists of more than two pairs of subdomains. Fig. 3.5 displays the same geometry, to the left with two adjacent segments along the common interface, and to the right further subdivided, with three subdomains along the common boundary curve in each Bézier patch.

It depends on the specific problem, which is considered, how far a geometry has to be subdivided. Especially for boundary conditions, it might be necessary to subdivide geometries, such that each patch consists of more than two adjacent pairs of segments along the common interface.

We apply algorithm 3.2 to different levels of subdivision, in order to see, which kind of basis functions of second kind are obtained in the case of degree (3, 3) Bézier patches.
First of all in Fig. 3.6 it can be seen, that for degree (3, 3) patches, all functions have local support. Furthermore different kinds of basis functions are obtained. To classify the functions in several types, can be beneficial if they are investigated in more detail.

- The coefficients of basis functions depicted in shades of blue, are defined on the lowest pair, or the lowermost two pairs of adjacent subdomains. There occur, irrespective of the level of subdivision, five of them.

- Those coefficients, which are shown in shades of red, refer to basis function with support on the top pair or the uppermost two two pairs of neighbouring segments along the common interface. If each Bézier patch is subdivided once, there are four
basis function of this kind. For every additional level of subdivision, five of these functions are obtained.

- The basis functions, with coefficients displayed in green, have support on three pairs of adjacent subdomains along the common boundary curve. They occur only if the Bézier patches are subdivided more than once. The total number of these basis functions is then $k - 2$.

- For geometries where each patch consists of four or more subdomains along the common interface, basis functions with support on four pairs of neighbouring segments occur, they are depicted in orange. There exist $k - 3$ of them.

Different shades of the same color are used, to make it easier to distinguish between neighbouring basis functions.

Although we already obtain basis functions, which are locally supported, we also want to consider Bézier patches of degree (4, 4). We will see, that the resulting basis functions have smaller support. This can be useful, for example for the calculation of explicit formulas for the coefficients of the basis functions, see [4].

### 3.3.3 Degree (4,4)

We investigate geometries consisting of Bézier patches, as displayed in Fig. 3.1, and further subdivided ones. One can already expect, that only basis functions with local support will be obtained, since this was already true for degree (3, 3) Bézier patches. Indeed Fig. 3.7 shows, that the basis of the null space consists of functions with local support only.

Fig. 3.7 displays the coefficients of basis functions of second kind for degree (4, 4) Bézier patches. Different types of functions are depicted in shades of distinct colors. Four types of functions exist, they are again categorized based on their support.

- Those basis functions, with support on the top pair or the first two pairs of segments along the common interface are displayed in shades of blue. Irrespective of the level
of subdivision, five of them are obtained.

- There always exist five basis functions, which are defined on the lowermost pair or the last two pairs of neighbouring subdomains along the common boundary curve. These functions are depicted in shades of red.

- The coefficients, which are displayed in shades of green, refer to basis functions, with support on two pairs of adjacent segments. There occur two different basis functions, which are then shifted downwards along the common boundary curve. In total there are $3k$ of them obtained.

- Depicted in yellow, are coefficients of basis functions, with support on three neighbouring subdomains. The number of such functions is always $k - 1$, they are only obtained, if the Bézier patch is more than once subdivided.
Bézier patches of higher degree will not be considered here. Instead we now turn our attention to geometries with more than two Bézier patches.
4 Basis for multi-patch domains

4.1 Three-patch domains

We consider geometries, which consist of three adjacent Bézier patches. All three meet in a common vertex, and there are three common boundary curves, each one for two Bézier patches respectively. We will not investigate Bézier patches of degree (2, 2), since we already know, that basis functions are not locally supported and that their number does not change with the number of inserted knot spans.

4.1.1 Degree (3, 3)

An example for a geometry, with Bézier patches of degree (3, 3), is displayed in Fig. 4.1.

In the three-patch case, and for all other multi-patch domains as well, we obtain three kinds of basis functions. Control points, which contribute to different kinds of basis functions, are depicted in different colors in Fig. 4.1.

1. Basis functions of first kind have their support on control points, which are displayed in black, i.e. the support is contained in one of the three patches only. They can be constructed similarly to the two-patch case, with use of equation (3.8).

2. Basis functions of second kind or edge basis function are supported by control points depicted in green. Those functions have their support on the common boundary curves and the neighbouring columns.
Figure 4.1: Three-patch geometry that consists of Bézier patches of degree (3,3). Control points, which have to be considered for basis functions of first kind, are displayed in black. The green control points are of interest for basis functions of second kind, those depicted in red for basis functions of third kind.

3. The control point in the vertex, and the surrounding control points, shown in red, contribute to basis functions of third kind or \textit{vertex basis functions}.

For the two-patch case, we concluded, that basis functions of first kind and basis functions of second kind can be investigated separately. For multi-patch geometries, only basis functions of first kind can be calculated without considering the other kinds of basis functions. It still holds, that the direct sum of the set of basis functions of first kind and the set of edge and vertex basis functions combined, forms the basis of $V$. Edge basis functions and vertex basis functions cannot be considered separately, since vertex basis functions have their support on control points shown in red and additionally some control points of the common interface and the neighbouring columns.

In order to apply algorithm 3.2 to the three-patch geometry, we need to order all control points. There are several ways to do this. In the following, for all multi-patch domains, a similar counter-clockwise ordering is applied. As an example, Fig. 4.2 shows the designation of control points for the degree (3,3) three-patch geometry. This kind of order makes it possible, to obtain similar structures in edge basis functions to basis functions of second kind in the two-patch case.
If we consider one boundary curve only, it is obvious that the six innermost control points now contribute to vertex basis functions. This is true for all three common interfaces. If we take a closer look at Fig. 3.6, we see, that independent of the level of subdivision, the first five basis functions have their support completely, or at least partly, on the first six control points. Hence, we can conclude, that we do not have to consider basis functions of second kind again, where those control points are equal to zero. Those basis functions,
denoted as edge basis functions, have a similar structure to the two-patch case.

**Edge basis functions**

Fig. 4.4 displays the edge basis functions for degree (3, 3) Bézier patches. It can be seen, that similar structures as in the two-patch case occur again. The difference comes from the fact, that only every third control point refers to one boundary curve and the neighbouring columns. Hence, there are two coefficients equal to zero in between, which correspond to control points on the other two common interfaces or their adjacent columns. Note that the colors in Fig. 4.4 correspond to the colors in Fig. 3.6.

Only edge basis functions for lower levels of subdivision are displayed, since it is obvious how further basis functions of second kind correspond to edge basis functions on multi-patch domains with higher levels of subdivision. In Fig. 4.4, the left picture depicts the edge basis functions for a geometry as in Fig. 4.2. The right one refers to the geometry in Fig. 4.3.
We already defined the different types of basis functions of second kind for degree (3, 3) Bézier patches. Edge basis functions can be characterized similarly, only the number of functions of some types differ. We give the total number of each type dependent on the valency of the geometry, since those results hold for all multi-patch domains, which consist of degree (3, 3) Bézier patches. A three-patch domain has valency 3, four-patch domains have valency 4 and so forth.

When a multi-patch geometry is considered, there occur three different types of vertices as depicted in Fig. 4.5. We denote interior vertices by $C_i$. Vertex basis functions are obtained around this kind of vertices. $Q_i$ refer to vertices, where two adjacent patches meet on the boundary. With $P_i$ we denote vertices on the boundary, which belong to one patch only.

1. There are $(4 \times \text{valency})$ edge basis functions in the vicinity of the $i$ boundary vertices $Q_i$. These functions have their support on one or two pairs of adjacent segments along the common boundary curve, the coefficients are depicted in red in Fig. 3.6 and Fig. 4.4.

2. We obtain two different types of edge basis functions in the interior of the edges.
a) The number of edge basis functions, which have their support on three pairs of neighbouring segments along the common interface is \((k - 1) \times \text{valency}\). They still occur only, if each Bézier patch is subdivided more than once. The coefficients of these functions are displayed in green in Fig. 3.6 and Fig. 4.4.

b) There exist \((k - 3) \times \text{valency}\) edge basis functions with support on four pairs of adjacent subdomains along the common boundary curve. Such functions are only obtained, if each Bézier patch consists of at least four segments along the common interface. The coefficients, which belong to this type of basis functions, are depicted in orange in Fig. 3.6.

Note, that \(k\) always denotes the number of knot spans, which were inserted per edge.

**Vertex basis functions**

Now, that we have detailed information about the edge basis functions, we finally want to consider the vertex basis functions. As mentioned before, the first five basis functions of second kind of each common interface merge into vertex basis functions. Note, that for the lowest level of subdivision one of those basis functions of second kind is different from all other levels of subdivision, see Fig. 3.6. Therefore we obtain slightly different vertex basis function for the lowest level of subdivision. In general there exist \((\text{valency} + 3)\) vertex basis functions. This results in six vertex basis functions for three-patch geometries. In Fig. 4.6 the vertex basis functions are displayed for a geometry, which consists of three degree \((3, 3)\) Bézier patches. The left picture refers to the lowest level of subdivision, the right figure depicts the basis functions for all other levels.

It can be seen, that the first three vertex basis functions are the same in both plots. The other three change if the geometry is subdivided more than once.
Figure 4.6: Vertex basis functions for three-patch geometry, which consists of degree (3, 3) Bézier patches. The picture to the left refers to geometries, which are subdivided once. The one to the right depicts the coefficients of basis functions for all higher levels of subdivision.

4.1.2 Degree \((4, 4)\)

Edge basis functions

Edge basis functions can again either be obtained using algorithm 3.2, or be constructed from basis functions of second kind of the two-patch geometry with degree \((4, 4)\) Bézier patches. We will not further investigate the exact structures of edge basis functions, since it was explained before, how they are related to basis functions of second kind. The total
numbers of different types of edge basis functions are as follows. These numbers hold for all multi-patch domains, with Bézier patches of degree (4, 4), and are therefore given with a dependency on the valency.

1. There exist \((5 \times \text{valency})\) edge basis functions in the vicinity of the \(i\) boundary vertices \(Q_i\), they have their support on the lowermost pair or the last two pairs of adjacent segments along the common interface.

2. There are two types of edge basis functions in the interior of the edges.
   a) We obtain \((3k \times \text{valency})\) edge basis functions, which are defined on two pairs of neighbouring segments along the common interface.
   b) The number of edge basis function, with support on three adjacent subdomains is \(((k - 1) \times \text{valency})\).

**Vertex basis functions**

Analogously to geometries, which consist of degree \((3, 3)\) Bézier patches, vertex basis functions have their support partly on the seven innermost control points. In other words, for each boundary curve, the first six control points now contribute to those basis functions. If we consider Fig. 3.7, we find, that this again concerns the first five basis functions of second kind. In contrast to degree \((3, 3)\) Bézier patches, these five basis functions of second kind are identical for all levels of subdivision in the case of degree \((4, 4)\) Bézier patches. Hence we obtain the same vertex basis functions for any level of subdivision. With use of algorithm 3.2, we can construct those vertex basis functions. The results are depicted in Fig. 4.7. Note, that we order the control points counter-clockwise, analogously to Fig. 4.2.

If we compare the vertex basis functions for three-patch geometries, which on the one hand consist of Bézier patches of degree \((4, 4)\), with those for Bézier patches of degree \((3, 3)\), we can see, that for higher degree, the basis functions have smaller support. This result is similar to two-patch geometries.
Figure 4.7: The vertex basis functions for a three-patch geometry, which consists of degree (4, 4) Bézier patches are displayed.

4.2 Four-patch domains

An arbitrary geometry can not be necessarily represented with use of two-patch and three-patch domains. Therefore we further consider four-patch geometries. As an example of such a geometry, Fig. 4.8 displays a four-patch geometry, which consists of degree (3, 3) Bézier patches, where each one is once subdivided.

Note, that the order of the control points is slightly different from the three-patch case. If an analogous strategy to the three-patch case is applied, algorithm 3.2 calculates all basis function, but for the vertex basis functions it does not result in a kind of banded structure. Therefore we use an ordering, as displayed in Fig. 4.8, and all presented results...
Figure 4.8: Four-patch geometry, with degree \((3, 3)\) Bézier patches. Control points, which refer to distinct kinds of basis functions, are depicted in different colors. The control points are ordered, such that algorithm 3.2 can be applied

are obtained with this assignment.

Similarly to the previous three-patch domains, we obtain three different kinds of basis functions for four-patch geometries. It was explained before, how to construct basis functions of first kind and edge basis functions. Hence we turn our attention to vertex basis functions. From Fig. 4.8, we know, that these basis functions have their support partly on the innermost nine control points.

**Degree \((3, 3)\)**

It was mentioned above, that there exist \((3 + \text{valency})\) vertex basis functions, Hence we obtain seven functions in the case of four-patch domains. We know as well, that for degree \((3, 3)\) Bézier patches, which are subdivided once, we get slightly different vertex basis functions, as if we subdivide more often. Fig. 4.9 displays the vertex basis functions for both cases.

It can be seen, that analogously to the three-patch case, the first three vertex basis functions, have the same structure independent of the level of subdivision.
Degree \((4, 4)\)

For four-patch geometries, which consist of degree \((4, 4)\) Bézier patches, we obtain the same seven vertex basis functions, independent of the level of subdivision. It can again be seen, that the basis functions for higher degree Bézier patches, have smaller support. The coefficients for the vertex basis functions are displayed in Fig. 4.10. It can also be seen, that we get a similar structure, to the three-patch case that is as banded as possible. As mentioned before, this is only possible, when ordering the control points similarly to Fig. 4.8.
4.3 Five-patch domains

The last domain we explicitly want to investigate is the five-patch geometry. An example of such a geometry, with degree \((3, 3)\) Bézier patches, is given in Fig. 4.11.

As in the four-patch case, we only take a closer look at the vertex basis functions. Hence only those functions, which have their support at least partly on the control points depicted in red. All following results are obtained with an ordering of the control points analogously to the four-patch case, see Fig. 4.8.
Degree (3, 3)

Just as in all other multi-patch domain cases, we obtain different vertex basis functions, depending on the level of subdivision. In Fig. 4.12 the coefficients of the vertex basis functions for degree (3, 3) Bézier patches are depicted. The picture to the left, displays the vertex basis functions for the case, that each Bézier patch is subdivided once. The figure to right refers to all other levels of subdivision.

Degree (4, 4)

For five-patch domains, with degree (4, 4) Bézier patches, we obtain the same eight vertex basis functions for any level of subdivision. Fig. 4.13 displays the coefficients of the vertex basis functions for such a geometry.

It can clearly be seen, that algorithm 3.2, can be applied to any kind of domain. For any degree \((p, p)\), with \(p > 2\), we obtain basis functions, which are locally supported. For higher degrees, the support of the basis functions is becoming smaller. Furthermore, independent of the choice of the order of the control points, all basis functions are calculated. If the
control points are ordered in a specific way, we also obtain a kind of banded structure in the resulting matrix $M$. 

Figure 4.12: Vertex basis functions for five-patch geometries, which consist of Bézier patches of degree $(3, 3)$. The left figure refers to the first level of subdivision, the right one to all further levels of subdivision.
Figure 4.13: The vertex basis functions for five-patch domains, with Bézier patches of degree (4, 4)
5 General bilinear multi-patch domains

Any arbitrary multi-patch geometry, which consists of more than five patches, can be represented by combining domains with fewer patches, which were described in detail in previous sections. The basis for such a geometry is obtained by suitably collecting basis functions of first kind as well as edge and vertex basis functions.

Throughout the first two sections of this chapter we consider as a representative example a seven-patch geometry, which is displayed in Fig. 5.1. It is represented by joining together a four-patch and a five-patch geometry. Therefore we obtain two inner vertices $C_1$ and $C_2$, seven vertices $Q_i$, where two adjacent patches meet on the boundary and five vertices $P_i$ on the boundary, which belong to one patch only.

In Fig. 5.1, we use different colors, to distinguish between control points, which belong
to one domain or both domains. The red area refers to the support of basis functions of the upper five-patch domain. The yellow subdomain contains the support of basis functions of the lower four-patch domain. The orange area belongs to both geometries, and is discussed in more detail in the following.

5.1 Basis functions of first kind and vertex basis functions

Basis functions of first kind are calculated with use of equation 3.8. For the outer patches, which are bounded by two black boundary curves and two inner edges, all displayed in yellow and red in Fig. 5.1, we obtain the same results as before. For the two inner patches, which are bounded by only one black boundary curve and three of the inner edges, we obtain less basis functions of first kind, since more control points contribute to edge and vertex basis functions. The parts of the two, which contain control points, that contribute to basis functions of first kind are depicted in light orange in Fig. 5.1.

Vertex basis functions for the two inner corner $C_1$ and $C_2$ are calculated with use of algorithm 3.2. Depending on the valency of the geometry the vertex belongs to, we obtain $(\text{valency} + 3)$ basis functions. It is necessary that sufficiently many knot spans are inserted along the edge, which connects these two vertices, such that vertex basis functions do not overlap.

5.2 Edge basis functions

We have to distinguish between the outer common boundary curves, which connect a vertex $C_i$ with an outer corner $Q_j$, depicted in blue and green in Fig. 5.1, and the inner edge, which is displayed in cyan and connects the two inner vertices $C_1$ and $C_2$. Edge basis functions, that have support on the outer edges and their adjacent columns, can be calculated without any further restrictions using algorithm 3.2.

The inner common interface requires additional considerations. The darker orange area
in Fig. 5.1 depicts the area, which contains the control points that have to be considered for these edge basis functions. Fig. 5.2 takes a closer look at this inner boundary curve.

As mentioned before, sufficiently many knot spans have to be inserted, in order to guarantee that vertex basis functions do not overlap. In Fig. 5.2 nine knot spans are inserted along the common boundary curve. We already know, that in the case of degree (4, 4) Bézier patches, the vertex basis functions have their support on two neighbouring segments along the common interface independent of the level of subdivision. All inner edge basis functions, i.e. those functions, which are not supported on the outermost six control points on both ends of the common boundary curve, have their support at most on three neighbouring segments along the common interface. Hence, we can conclude, that there have to be at least three knot spans, whose control points do not contribute to vertex basis functions. For (3, 3) Bézier patches, the vertex basis functions have their support on three adjacent segments along the common edge, if the patches are more than once subdivided. The inner edge basis functions have their support on up to four adjacent segments along the common edge. Therefore we need at least four knot spans in between those, which at least partly contribute to vertex basis functions. Fig. 5.2 refers to a seven-patch domain, which consists of Bézier patches of degree (4, 4). The knot spans, which contain control
points, that contribute to vertex basis functions, are displayed in red (there are two of them on each end of the common edge), all other knot spans are depicted in cyan (five of these occur in this figure).

5.3 The general case

In the case of a general domain, we finally obtain the following for degree \((3, 3)\) Bézier patches

1. \((\text{valency} + 3)\) vertex basis functions, for each inner vertex \(C_i\),

2. \((5 + (k - 2) + (k - 3))\) edge basis functions, for every common interface except of those between inner vertices and

3. \(((k - 2) + (k - 3))\) edge basis functions for those boundary curves, which connect two interior vertices.

In the case of Bézier patches with degree \((4, 4)\) there exist for general multi-patch domains

1. \((\text{valency} + 3)\) vertex basis functions for every interior vertex \(C_i\),

2. \((5 + 3k + (k - 1))\) edge basis functions for each common boundary curve, which does not connect inner vertices and

3. \((3k + (k - 1))\) edge basis functions for each common interface between two inner vertices.
6 Conclusion

We studied the basis of the coefficient space of $C^1$-smooth geometrically continuous isogeometric functions. We considered bilinearly parameterized geometries and introduced a new algorithm, with which the coefficients of these basis functions can be calculated.

The algorithm was then applied to different two-patch domains, which consist of degree $(d, d)$ Bézier patches. For $d = 2$ we obtained globally supported basis functions only. Hence we considered Bézier patches of higher degree. For $d = 3, 4$ basis functions with local support were obtained. Thereafter we studied three-patch, four-patch and five-patch domains for $d = 3$ and $d = 4$. We also demonstrated how to represent a general bilinear multi-patch domain, which consists of more than five Bézier patches, by joining together domains, which contain fewer patches.

In [4], Poisson’s equation and the biharmonic equation have already been solved, with use of $C^1$-smooth geometrically continuous isogeometric functions. In future, the introduced algorithm could be used to solve further partial differential equations. Furthermore, only two-dimensional domains have been considered so far. It could be interesting to study volumes and geometrically continuous isogeometric functions on such three-dimensional domains.
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Bibliography


