This is a preprint of an article accepted for publication in A. K. Belyaev, V. V. Eliseev, E. A. Oborin. About One-dimensional Models for Describing Elastic Microslip in Belt Drive. International Review of Mechanical Engineering (I.R.E.M.E.) 10(5): 333–338. DOI 10.15866/ireme.v10i5.8944 Copyright © 2016 Praise Worthy Prize S.r.l. - All rights reserved

About One-dimensional Models for Describing Elastic Microslip in Belt Drive

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Abstract – The stress-strain state of a transmitting belt as an elastic rod with the transversal shear is considered. Using the variational method we derive equations of a new one-dimensional linear theory. The solution with the effect of elastic microslip is obtained. The transition boundaries to the partial slip and to the gross slip are determined.

Keywords: Belt drive, Elastic microslip, Variational method, Rod model, Transverse shear

I. Introduction

The belt drive design in engineering practice is built upon a basic comprehension about kinematics and force distribution (see e.g. [1]). In the case of high speeds and loads this comprehension relies on the elastic extensible string model [2]–[5]. However the contradictions related to the contact force distribution still remain.

To gain a better understanding of the idea of this paper one needs a more thorough insight into the belt-pulley contact. Early examples of research in this direction include works of Reynolds [6] and Zhukovsky [7]. They pointed out the phenomena of elastic microslip; the matter of belt transmission is unclear without it. One should note recent works [8], [9] as valuable extensions. In [8] the approach from [1] with account for tangential and normal acceleration components is presented. It is supposed there that slip is absent at the point where the belt enters the pulley. Steady dynamics of belt drive on a number of pulleys is developed in [9]. The area of slip is supposed, and the Coulomb friction law is used.

In the papers [10]–[13] a new approach is developed taking into account shear deformation. In [10] one finds the basics of power transmission in belt drives by means of shear. In [11] the comparison of two approaches is done, their combination is reported in [12]. The effect of sliding friction in the pulley grooves of V-belt transmissions is proposed in [13].

Finite element models are widely used in [14]–[16]. The detailed examinations of the friction between bodies are given in [17] taking into account elementary models of deformation. A review of the friction models and their applications is presented in [18].

In the present work the topic of elastic microslip is considered by using more sophisticated belt models. The

goal of this work is to explain the paradoxes on the elementary scheme having a perspective to create a new design method. Therefore the current study addresses just the static case, and dynamic simulations of the phenomena related to the friction (see for discussion of dynamic effects [19], [20]) remain for further investigations.

II. Construction of model by variational method

A very simple scheme of the belt drive is shown in Fig. 1. Assuming rigid plates, the displacement U_1 of the driving plate is prescribed and transmitted to the driven one by the belt. The interaction between the plates and the belt is done by friction force. The resistance force F acts on the driven plate which has the displacement U_2 . The unit volume load is directed along x-axis: $\mathbf{f} = f \mathbf{i}$ where $\mathbf{i}$ is the orthonormal unit vector of this axis.

Fig. 1. Simple belt drive scheme

In the applied studies the variational method based on the principle of minimum potential energy is effective for the constructing of one-dimensional models [21]. One uses a displacement approximation over the cross section in this method.

We take the expressions of the displacement projections on the Cartesian axes with two varied functions $u(x), \gamma(x)$ in the following form:

$$\begin{aligned} u_x &= u(x) + \gamma(x)y, \\ u_y &= u_z = 0. \end{aligned} \quad (1)$$

Here u is the axial displacement of the belt; it differs from the plate displacements $U = u - \gamma h$ where h is the distance between the x-axis and the plate (Fig. 1). The difference $u - U = \gamma h$ may be interpreted as elastic microslip. However if $u_x|_{y=-h} \neq U$ then we understand it as the partial slip or the gross (or macro) slip in the usual sense.

Using the displacements (1) we find the strains

$$\begin{aligned} \varepsilon_x &= u' + \gamma'y, \gamma_{xy} = \gamma, \\ \varepsilon_y &= \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \quad (2)$$

and calculate the energy per unit length:

$$2\Pi = \int_{\Omega} (E\varepsilon_x^2 + \mu\gamma_{xy}^2) d\Omega = E(\Omega u'^2 + J\gamma'^2) + \mu\Omega\gamma^2. \quad (3)$$

Here E is the Young's modulus, μ is the shear modulus; $J = \int y^2 d\Omega$ is the moment of inertia, namely the integral over the cross section with the area Ω . The x-axis is assumed to pass through the mass centers of sections, therefore $\int y d\Omega = 0$.

Then we use the principle of virtual work:

$$\int_0^L (\delta\Pi - \int f \delta u_x d\Omega) dx = 0 \quad (4)$$

where L is the belt length before the deformation. The loads are thought of as body forces, and the contact (surface) forces are the limiting case of them.

The variational formulation (4) yields the system of two ordinary differential equations (ODE):

$$\begin{aligned} T' + p_u &= 0, \\ P' - \frac{\partial\Pi}{\partial\gamma} + p_\gamma &= 0; \\ T &\equiv \frac{\partial\Pi}{\partial u'} = E\Omega u', P \equiv \frac{\partial\Pi}{\partial\gamma'} = EJ\gamma'. \end{aligned} \quad (5)$$

Obviously, T is the belt tension force, p_u is the distributed axial load. The first equation in (5) is the force balance, the second is the relation between the force factor P , the load p_γ , and the elastic microslip γ . The

loads can be expressed as integrals over the cross section using the body force f :

$$\begin{aligned} p_u &\equiv \int f d\Omega, \\ p_\gamma &\equiv \int f y d\Omega. \end{aligned} \quad (6)$$

The variational equation (3) yields also the boundary conditions. We assume $T = 0, P = 0$ at the free and unloaded ends.

Two linear ODE with the constant coefficients in (5) admit analytical solution.

III. Contact problem

The plates exert a surface friction force $t(x)$ upon the belt at two places, namely, at the first plate to the left, at the second plate to the right:

$$\begin{aligned} 0 < x < l_2 : p_u &= -t, \\ p_\gamma &= th; \\ x_1 < x < L : p_u &= t, \\ p_\gamma &= -th. \end{aligned} \quad (7)$$

Here the coordinate is $x_1 = L - l_1$. The function $t(x)$ is unknown, but the displacement of the driver plate

$$x_1 < x < L : u - \gamma h = U_1 \quad (8)$$

and the resistance force

$$\int_0^{l_2} t dx = F \quad (9)$$

are prescribed. For the driven plate we have:

$$0 < x < l_2 : u - \gamma h = U_2. \quad (10)$$

The unknown constant U_2 can be determined using the force balance equation on the free span and the relation of elasticity (5):

$$l_2 \leq x \leq x_1 : T = \text{const} = F = E\Omega \frac{U_1 - U_2}{x_1 - l_2}. \quad (11)$$

We make use of the relation between the loads on the contact segments

$$hp_u + p_\gamma = 0 \quad (12)$$

and derive the following equation from (5):

$$(hT + P)' = \frac{\partial \Pi}{\partial \gamma}. \quad (13)$$

Hence with the derivative $u' = h\gamma'$ of elastic microslip we write the equation

$$\begin{aligned} \gamma'' - \lambda^2 \gamma &= 0, \\ \lambda &\equiv \sqrt{\frac{\mu \Omega}{E(\Omega h^2 + J)}}. \end{aligned} \quad (14)$$

And the general solution of (14) is

$$\gamma = Ae^{\lambda x} + Be^{-\lambda x}. \quad (15)$$

Since the segments are divided, two pairs of constants A, B arise.

Other equations are valid on the segment between the plates. Due to absence of loads $p_u = 0, p_\gamma = 0$ from (5) we have

$$T = \text{const}, P' = \frac{\partial \Pi}{\partial \gamma}. \quad (16)$$

Using (5) again we write

$$\gamma'' - \Lambda^2 \gamma = 0, \Lambda = \sqrt{\frac{\mu \Omega}{EJ}}. \quad (17)$$

Similarly to (15) the general solution of (17) is

$$\gamma = Ce^{\Lambda x} + De^{-\Lambda x}. \quad (18)$$

We have four boundary conditions for the tension force (it is continuous):

$$\begin{aligned} x = 0, L : T = 0 &\Rightarrow \gamma' = 0; \\ x = l_2, x_1 : T = F &= E\Omega h \gamma'. \end{aligned} \quad (19)$$

Hence we obtain the solution in terms of elastic microslip $\gamma(x)$:

$$\begin{aligned} x < l_2 : \gamma &= \frac{\gamma'(l_2) \cosh \lambda x}{\lambda \sinh \lambda l_2}; \\ x > x_1 : \gamma &= \frac{\gamma'(x_1) \cosh \lambda (x - L)}{\lambda \sinh \lambda (x_1 - L)}; \\ l_2 < x < x_1 : \gamma &= \frac{\gamma(l_2) \sinh \Lambda (x - x_1) - \gamma(x_1) \sinh \Lambda (x - l_2)}{\sinh \Lambda (l_2 - x_1)}; \\ \gamma'(l_2) = \gamma'(x_1) &= F / E\Omega h. \end{aligned} \quad (20)$$

At the plates the tension T is proportional to γ' , cf. (19). The displacement is $u(x) = U_{1,2} + \gamma h$.

IV. Slip

The product γh is equal to the displacement difference between the belt and the plate and can be thought of as the elastic microslip value [17]. Herewith the absence of slip (on the side $y = -h$) can be ensured only until the friction force hits the dry friction limit

$$t = E\Omega h \gamma'' = E\Omega h \lambda^2 \gamma < t_*. \quad (21)$$

This equality is met at the left plate whereas a minus sign appears at the right plate. We need to construct a solution for which $t = t_*$ at the places where condition (21) is violated.

The partial slip begins from the inner parts of the plates, because t is maximal there. At the driven plate and at the driver plate it starts under the following loads, correspondingly:

$$\begin{aligned} F &= F_{2*} = t_* \tanh \lambda l_2 / \lambda; \\ F &= F_{1*} = t_* \tanh \lambda l_1 / \lambda. \end{aligned} \quad (22)$$

Since the function $\varphi(x) = \tanh \gamma x / \lambda$ increases, the slip starts from the shorter plate (from the driven one if $l_1 < l_2$).

Let us consider the mode of partial slip ($F > F_*$). The segment occupied by the plate divides into two parts: to the left of the point $x = x_*$ the slip is not present yet (but the elastic microslip is), and to the right of this point it appears (see Fig. 2).

Fig. 2. Elastic microslip

We have the following relations on the parts:

$$\begin{aligned}
 \underline{x < x_*} : \quad & u' = h\gamma', \\
 & \gamma = A \cosh \lambda x, \\
 & T = AE\Omega h \lambda \sinh \lambda x, \\
 & t = E\Omega h \lambda^2 \gamma, \\
 \underline{x > x_*} : \quad & T' = t_* \Rightarrow T = F + t_*(x - l_2), \\
 & \gamma = B \cosh \lambda x + C \sinh \lambda x + t_* h / \mu \Omega.
 \end{aligned} \quad (23)$$

At the point between them we match the solutions using the continuity of the tension force and of the friction load:

$$\underline{x = x_*} : \quad \begin{cases} AE\Omega h \lambda \sinh \lambda x_* = F + t_*(x_* - l_2), \\ AE\Omega h \lambda^2 \cosh \lambda x_* = t_*. \end{cases} \quad (24)$$

The last equations determine the force $F(x_*)$ as a function of x_* in the following form:

$$F = t_* \left(\frac{\tanh \lambda x_*}{\lambda} - x_* + l_2 \right). \quad (25)$$

The limiting value $F(0) = t_* l_2$ is the drive loading capacity, i.e. it is impossible to exceed this load.

V. Numerical results

Figs. 2–4 display the results of the calculation $\gamma(x)$, $T(x)$, $t(x)$ for the following set of parameters (in SI units):

$$\begin{aligned}
 L = 1, l_1 = 0.15, l_2 = 0.1, E = 10^7, \mu = E/3, \\
 h = 0.03, u_1 = 0.01, F = 10, t_* = 150.
 \end{aligned}$$

A square cross section of the belt is assumed.

Fig. 3. Elastic microslip

In Fig. 3 we see the localization of $\gamma(x)$ (for the solution (20) without slip) at the inner boundaries of the

plates. With transition to the free span ($l_2 < x < x_1$) the effect disappears, but not abruptly.

Fig. 4. Belt tension.
Dashed line for partial slip solution

As expected the tension is constant outside the plates (see Fig. 4). And at the plates inner boundaries we observe the sharp decrease of tension. At the ends of the belt, it becomes zero.

Fig. 5. Friction force.
Dashed line for partial slip solution

The friction force is concentrated at the inner boundaries of the plates (see Fig. 5).

Fig. 6. Function for force calculation

In Fig. 6 the graph of function $F(x_*)$ is plotted using (25) for the left driven plate. The loading capacity is exhausted with the force $F(0) = t_* l_2$.

In numerical work we obtain $F_{*1} = F_{*2} = F(l_2) = 3$, $F(0) = 15$ (see Fig. 6). The load given above ($F = 10$) is large enough to start the partial slip, therefore the first solution (see Sect. III) without partial slip exceeds the boundaries of its applicability. The coordinate of the slip zone boundary is $x_* = 0.053$.

In Fig. 6 we see the dependence of the given force F on the difference $U_1 - U_2$ of plate displacements. When the value $F_{*1} = 3$ is exceeded, the graph of the solution with partial slip (dashed line) begins to deviate from the straight line. The partial slip zone expands causing nonlinearity.

To obtain the displacement and to plot the graph in Fig. 6 one needs to inverse the function $F(x_*)$ numerically and determine the coefficient A from (24)

$$A = \frac{F + t_*(x_*(F) - l_2)}{E\Omega h \sinh \lambda x_*(F)} \quad (26)$$

and similarly at the driver plate.

Fig. 6. Force-displacement curve.
Dashed line for partial slip solution

VI. Timoshenko beam

The model with approximation (1) seems to be unusual however it is a certain representation of a beam with shear. Let us take use of the Timoshenko beam whose equations are given by:

$$\begin{aligned} Q' + q &= 0, \\ M' + Q + m &= 0, \\ M &= a\theta', \\ Q &= b(v' - \theta). \end{aligned} \quad (27)$$

Here Q, M are the transverse force and the bending moment, q, m are the distributed force and moment, θ, v are the rotation angle and the deflection, a, b are the bending stiffness and the shear stiffness, respectively. The equations can be derived by the variational method with the approximation

$$\begin{aligned} u_x &= -\theta(x)y, \\ u_y &= v(x). \end{aligned}$$

Without deflection ($v = 0$) we obtain from (27):

$$a\theta'' - b\theta + m = 0. \quad (28)$$

However this coincides with the equations (14), (17) for γ , since

$$a = EJ, b = \mu\Omega, m = p_\gamma.$$

The model with nonhomogeneous tension over the area is the model with extension and shear without deflection. The contact load on the belt reduces not only to the forces, but also to the moments.

The established coincidence allows us to use subsequently the well-developed theory of rods and the method of solving even the nonlinear problems [22] for the further study of belt drive mechanics.

VII. Conclusion

The simplified model proposed in this study allows one to understand the phenomenon of elastic microslip completely. Also the transitions to the partial slip and to the gross slip are clarified. It is becoming clear that the accurate calculation of the belt drive can be done only with account for the nonlinear model of Cosserat-Timoshenko rod.

Acknowledgements

This research is carried out in the framework of the joint project of the Russian Foundation for Basic Research (grant No. 14-51-15001) and the Austrian Science Fund (FWF, grant No. I 2093 International Project).

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