Orthogonality preserving transformations of Hilbert Grassmannians

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Abstract

Let $H$ be a complex Hilbert space and let $G_k(H)$ be the Grassmannian formed by $k$-dimensional subspaces of $H$. Suppose that $\dim H > 2k$ and $f$ is an orthogonality preserving injective transformation of $G_k(H)$, i.e. for any orthogonal $X, Y \in G_k(H)$ the images $f(X), f(Y)$ are orthogonal. Furthermore, if $\dim H = n$ is finite, then $n = mk + i$ for some integers $m \geq 2$ and $i \in \{0, 1, \ldots, k - 1\}$ (for $i = 0$ we have $m \geq 3$). We show that $f$ is a bijection induced by a unitary or anti-unitary operator if $i \in \{0, 1, 2, 3\}$ or $m \geq i + 1 \geq 5$; in particular, the statement holds for $k \in \{1, 2, 3, 4\}$ and, if $k \geq 5$, then there are precisely $(k - 4)(k + 1)/2$ values of $n$ such that the above condition is not satisfied.

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1. Introduction

By Gleason’s theorem [2], the set of pure states of a quantum mechanical system can be identified with the set of rank-one projections, i.e. the set of rays in a complex Hilbert space. Wigner’s theorem [12] describes symmetries of quantum mechanical systems, it states that every bijective transformation of the set of pure states preserving the tran-
sition probability (the trace of the composition of two projections or, equivalently, the angle between two rays) is induced by a unitary or anti-unitary operator. Also, there is a non-bijective version of this result concerning linear and conjugate-linear isometries. Various kinds of Wigner-type theorems can be found, for example, in [7]; some of them are formulated in terms of orthogonality preserving transformations.

Let $H$ be a complex Hilbert space. For every positive integer $k < \dim H$ we denote by $\mathcal{G}_k(H)$ the Grassmannian formed by $k$-dimensional subspaces of $H$. This Grassmannian can be naturally identified with the set of rank-$k$ projections. In the case when $\dim H \geq 2k$, two $k$-dimensional subspaces are orthogonal if and only if the composition of the corresponding projections is zero.

Suppose that $\dim H \geq 3$. Then the bijective version of Wigner’s theorem is a consequence of the following Uhlhorn’s observation [10]: every bijective transformation of $\mathcal{G}_1(H)$ preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator. In fact, the latter statement follows fairly easily from the Fundamental Theorem of Projective Geometry ([7, Proposition 4.8] or [11, Theorem 4.29]). See [8] for a more general approach to orthogonality preserving transformations.

Uhlhorn’s observation was extended on other Grassmannians by Győry [3] and Šemrl [9]: if $\dim H > 2k$, then every bijective transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions is induced by a unitary or anti-unitary operator. A simple example shows that the statement fails for $\dim H = 2k$. If $H$ is infinite-dimensional, then the same holds for orthogonality preserving (in both directions) bijective transformations of the Grassmannian formed by subspaces whose dimension and codimension both are infinite [9]. Győry–Šemrl’s theorem is used to study transformations preserving the gap metric [1] and commutativity preserving transformations [4,6]. The assumption of surjectivity cannot be omitted. It was noted in [9] that for the case when $H$ is infinite-dimensional there are non-surjective transformations of $\mathcal{G}_k(H)$ which are not induced by linear or conjugate-linear isometries and preserve the orthogonality relation in both directions. If $H$ is finite-dimensional and $\dim H > 2k$, then such transformations do not exist, i.e. every transformation of $\mathcal{G}_k(H)$ preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator [5]. In this note, we obtain an analogue of the latter result for injective transformations under the assumption that the orthogonality relation is preserved exactly in one direction.

2. Result

Suppose that $H$ is finite-dimensional. Let $k$ be a positive integer such that $\dim H > 2k$. Then

$$\dim H = mk + i$$

for some integers $m \geq 2$ and $i \in \{0, 1, \ldots, k - 1\}$. Note that $m \geq 3$ if $i = 0$.

Theorem 1. Suppose that one of the following conditions is satisfied:

• \( i \in \{0, 1, 2, 3\} \);
• \( i \geq 4 \) and \( m \geq i + 1 \).

Then every injective transformation \( f \) of \( G_k(H) \) preserving the orthogonality relation, i.e. for any orthogonal \( X, Y \in G_k(H) \) the images \( f(X), f(Y) \) are orthogonal, is a bijection induced by a unitary or anti-unitary operator on \( H \).

Theorem 1 shows that every injective transformation of \( G_k(H) \) preserving the orthogonality relation is a bijection induced by a unitary or anti-unitary operator if

• \( k \in \{1, 2, 3, 4\} \) or
• \( k \geq 5 \) and \( \dim H \) is sufficiently large, for example, if \( \dim H \geq k^2 \).

For \( k \geq 5 \) there are precisely
\[
3 + 4 + \cdots + (k - 2) = (k - 4)(k + 1)/2
\]
values of \( \dim H \) such that the condition of Theorem 1 does not hold (for every \( i \in \{4, \ldots, k - 1\} \) the corresponding values of \( m \) are \( 2, 3, \ldots, i \)).

If \( k = 1 \), then Theorem 1 is a simple consequence of the Fundamental Theorem of Projective Geometry [7, Remark 49].

3. Proof of Theorem 1

First of all, we present some facts which will be exploited to prove Theorem 1. Two \( k \)-dimensional subspaces of \( H \) are called adjacent if their intersection is \( (k - 1) \)-dimensional or, equivalently, their sum is \( (k + 1) \)-dimensional. Any two distinct 1-dimensional subspaces of \( H \) are adjacent. Similarly, if \( \dim H = n \) is finite, then any two distinct \( (n - 1) \)-dimensional subspaces of \( H \) are adjacent. If \( X, Y \in G_k(H) \), then the distance \( d(X, Y) \) between \( X \) and \( Y \) is defined as the smallest integer \( m \) such that there is a sequence
\[
X = X_0, X_1, \ldots, X_m = Y,
\]
where \( X_{j-1}, X_j \) are adjacent elements of \( G_k(H) \) for every \( j \in \{1, \ldots, m\} \). It is well known that
\[
d(X, Y) = k - \dim(X \cap Y) = \dim(X + Y) - k,
\]
see, for example, [7, Section 2.3].

Theorem 2 ([5]). Suppose that \( \dim H > 2k > 2 \) and \( f \) is a transformation of \( G_k(H) \) satisfying the following conditions:
• $f$ is adjacency preserving, i.e. for any adjacent $X, Y \in G_k(H)$ the images $f(X), f(Y)$ are adjacent;
• $f$ is orthogonality preserving.

Then $f$ is induced by a linear or a conjugate-linear isometry.

Also, we will need the following result mentioned in the Introduction.

**Theorem 3 ([5]).** If the dimension of $H$ is finite and greater than $2k$, then every transformation of $G_k(H)$ preserving the orthogonality relation in both directions is a bijection induced by a unitary or anti-unitary operator on $H$.

Suppose that $\dim H = n$ is finite. It was noted above that Theorem 1 holds for $k = 1$. We assume that $k \geq 2$ and

$$n = mk + i > 2k,$$

where $m \geq 2$ and $i \in \{0, 1, \ldots, k - 1\}$. Let $f$ be an injective transformation of $G_k(H)$ preserving the orthogonality relation.

### 3.1. The case $i = 0$

In this case, we have $m \geq 3$. Suppose that $f(X), f(Y)$ are orthogonal for some $X, Y \in G_k(H)$; we show that $X, Y$ are orthogonal.

Observe that

$$\dim (X^\perp \cap Y^\perp) \geq n - 2k = (m - 2)k.$$

In the case when $m \geq 4$, there are mutually orthogonal $k$-dimensional subspaces

$$Z_1, \ldots, Z_{m - 2} \subset X^\perp \cap Y^\perp;$$

if $m = 3$, then we take any $k$-dimensional subspace $Z_1$ in $X^\perp \cap Y^\perp$. Since $f$ is orthogonality preserving, each $f(Z_j)$ is orthogonal to $f(X) + f(Y)$. By our assumption, $f(X)$ and $f(Y)$ are orthogonal. The dimension of $H$ is equal to $mk$ and $H$ is the orthogonal sum of

$$f(Z_1), \ldots, f(Z_{m - 2}), f(X), f(Y).$$

The dimension of $X^\perp$ is equal to $n - k = (m - 1)k$, i.e. $X^\perp$ contains the unique $k$-dimensional subspace $Z$ orthogonal to all $Z_j$ and $H$ is the orthogonal sum of

$$f(Z_1), \ldots, f(Z_{m - 2}), f(X), f(Z).$$
Therefore, \( f(Y) = f(Z) \). Since \( f \) is injective, we have \( Y = Z \), i.e. \( Y \) is orthogonal to \( X \).

So, \( f \) is orthogonality preserving in both directions and Theorem 3 gives the claim.

### 3.2. The case \( i \in \{1, 2, 3\} \)

It is sufficient to show that \( f \) is adjacency preserving and to apply Theorem 2.

The general case can be reduced to the case when \( m = 2 \). If \( X, Y \in \mathcal{G}_k(H) \) are adjacent and \( m \geq 3 \), then we take mutually orthogonal \( X_1, \ldots, X_{m-2} \in \mathcal{G}_k(H) \) whose sum is orthogonal to \( X + Y \) (for \( m = 3 \) we choose any \( X_1 \in \mathcal{G}_k(H) \) orthogonal to \( X + Y \)). Consider the \((2k + i)\)-dimensional subspaces \( M \) and \( N \) which are the orthogonal complements of

\[
X_1 + \cdots + X_{m-2} \quad \text{and} \quad f(X_1) + \cdots + f(X_{m-2}),
\]

respectively. Then \( f \) transfers any \( k \)-dimensional subspace of \( M \) to a \( k \)-dimensional subspace of \( N \), i.e. it induces an orthogonality preserving injection of \( \mathcal{G}_k(M) \) to \( \mathcal{G}_k(N) \).

From this moment, we assume that \( m = 2 \), i.e. \( n = 2k + i \) with \( i \in \{1, 2, 3\} \). Let \( \mathcal{G} \) be the set of all subspaces \( U \subset H \) satisfying

\[
k < \dim U \leq k + i.
\]

For every \( U \in \mathcal{G} \) we define \( f'(U) \) as the smallest subspace containing \( f(Y) \) for all \( k \)-dimensional subspaces \( Y \subset U \). Since \( f \) is injective, \( \dim f'(U) > k \). If \( Z \) is a \( k \)-dimensional subspace orthogonal to \( U \) (the inequality \( \dim U^\perp \geq k \) guarantees that such a subspace exists), then \( f'(U) \) is orthogonal to \( f(Z) \) which implies that \( \dim f'(U) \leq k + i \) and \( f'(U) \) belongs to \( \mathcal{G} \). So, \( f' \) is a transformation of \( \mathcal{G} \) and for every \( U \in \mathcal{G} \) we have

\[
f(\mathcal{G}_k(U)) \subset \mathcal{G}_k(f'(U)).
\]

If \( U, S \in \mathcal{G} \) are orthogonal, then the same holds for \( f'(U), f'(S) \).

The cases \( i = 1 \) and \( i = 2 \) are simple.

**Lemma 1.** If \( i \in \{1, 2\} \), then \( f \) is adjacency preserving.

**Proof.** If \( i = 1 \), then \( \mathcal{G} = \mathcal{G}_{k+1}(H) \) and we have

\[
f'(\mathcal{G}_{k+1}(H)) \subset \mathcal{G}_{k+1}(H)
\]

which implies that \( f \) is adjacency preserving (since \( X, Y \in \mathcal{G}_k(H) \) are adjacent if and only if \( X + Y \in \mathcal{G}_{k+1}(H) \)).

Let \( i = 2 \). Then

\[
\mathcal{G} = \mathcal{G}_{k+1}(H) \cup \mathcal{G}_{k+2}(H).
\]
If \( f \) is not adjacency preserving, then there is \( U \in \mathcal{G}_{k+1}(H) \) such that \( f'(U) \) is an element of \( \mathcal{G}_{k+2}(H) \). Note that \( U^{\perp} \) belongs to \( \mathcal{G}_{k+1}(H) \) and \( f'(U), f'(U^{\perp}) \) are orthogonal. This implies that the dimension of \( f'(U^{\perp}) \) is not greater than \( k \) which is impossible. \( \Box \)

The case \( i = 3 \) is more complicated and the proof will be given in several steps. In this case, \( \mathcal{G} \) is the union of \( \mathcal{G}_{k+1}(H), \mathcal{G}_{k+2}(H) \) and \( \mathcal{G}_{k+3}(H) \). It is sufficient to establish that

\[
f'(\mathcal{G}_{k+2}(H)) \subset \mathcal{G}_{k+2}(H).
\]

Indeed, if \( U \in \mathcal{G}_{k+1}(H) \), then \( U^{\perp} \) belongs to \( \mathcal{G}_{k+2}(H) \) and the latter inclusion shows that the same holds for \( f'(U^{\perp}) \). Since \( f'(U) \) and \( f'(U^{\perp}) \) are orthogonal, the dimension of \( f'(U) \) is not greater than \( k + 1 \) which means that \( f'(U) \in \mathcal{G}_{k+1}(H) \). So, \( f' \) transfers \( \mathcal{G}_{k+1}(H) \) to itself and \( f \) is adjacency preserving.

Our first step is to show that

\[
f'(\mathcal{G}_{k+2}(H)) \subset \mathcal{G}_{k+1}(H) \cup \mathcal{G}_{k+2}(H).
\]

If \( U \in \mathcal{G}_{k+2}(H) \), then \( U^{\perp} \in \mathcal{G}_{k+1}(H) \) and \( f'(U), f'(U^{\perp}) \) are orthogonal. In the case when \( f'(U) \) belongs to \( \mathcal{G}_{k+3}(H) \), the dimension of \( f'(U^{\perp}) \) is not greater than \( k \), a contradiction.

Two distinct elements of \( \mathcal{G}_{k+1}(H) \cup \mathcal{G}_{k+2}(H) \) are said to be \('\)-adjacent if one of the following possibilities is realized:

- these are adjacent elements of \( \mathcal{G}_{k+2}(H) \);
- one of them belongs to \( \mathcal{G}_{k+2}(H) \), the other to \( \mathcal{G}_{k+1}(H) \) and the \((k+1)\)-dimensional subspace is contained in the \((k+2)\)-dimensional.

**Lemma 2.** If \( U, V \in \mathcal{G}_{k+2}(H) \) are adjacent, then \( f'(U), f'(V) \) are \('\)-adjacent or \( f'(U) = f'(V) \).

**Proof.** The subspace \( U \cap V \) is \((k+1)\)-dimensional and contains infinitely many \( k \)-dimensional subspaces. Therefore, the subspace \( f'(U) \cap f'(V) \) contains infinitely many \( k \)-dimensional subspaces which is possible only in the case when \( f'(U), f'(V) \) are \('\)-adjacent or coincident. \( \Box \)

The following lemma completes the proof for the case \( i = 3 \).

**Lemma 3.** \( f'(U) \in \mathcal{G}_{k+2}(H) \) for every \( U \in \mathcal{G}_{k+2}(H) \).

**Proof.** Recall that two subspaces \( U, V \subset H \) are called compatible if there are subspaces \( U' \subset U \) and \( V' \subset V \) such that \( U \cap V, U', V' \) are mutually orthogonal and

\[
U = U' + (U \cap V), \quad V = V' + (U \cap V).
\]
Let $U$ and $V$ be distinct $(k + 2)$-dimensional subspaces of $H$. Then $\dim(U \cap V) \geq 1$. The following two conditions are equivalent:

- $U, V$ are compatible and $\dim(U \cap V) = 1$;
- there are infinitely many $k$-dimensional subspaces of $U$ which are orthogonal to infinitely many $k$-dimensional subspaces of $V$.

Suppose that one of these conditions holds. Then there are infinitely many $k$-dimensional subspaces of $f'(U)$ which are orthogonal to infinitely many $k$-dimensional subspaces of $f'(V)$.

The equation $\dim(U \cap V) = 1$ shows that $d(U, V) = k + 1$ and there is a sequence

$$U = U_0, U_1, \ldots, U_k, U_{k+1} = V,$$

where $U_j, U_{j+1}$ are adjacent elements of $\mathcal{G}_{k+2}(H)$ for every $j \in \{0, 1, \ldots, k\}$. By Lemma 2,

$$f'(U_j), f'(U_{j+1}) \in \mathcal{G}_{k+2}(H) \cup \mathcal{G}_{k+1}(H)$$

are $t$-adjacent or coincident; in particular, if for a certain $j \in \{0, 1, \ldots, k\}$ both $f'(U_j), f'(U_{j+1})$ belong to $\mathcal{G}_{k+1}(H)$, then $f'(U_j) = f'(U_{j+1})$.

Suppose that at least one of $f'(U)$, $f'(V)$, say $f'(V)$, belongs to $\mathcal{G}_{k+1}(H)$. Since there are infinitely many $k$-dimensional subspaces of $f'(U)$ which are orthogonal to infinitely many $k$-dimensional subspaces of $f'(V)$, one of the following possibilities is realized:

1. $f'(U) \in \mathcal{G}_{k+2}(H)$ is the orthogonal complement of $f'(V)$;
2. $f'(U), f'(V) \in \mathcal{G}_{k+1}(H)$ are orthogonal.

In the case (1), we take the maximal $j \in \{0, 1, \ldots, k + 1\}$ such that $f'(U_j)$ belongs to $\mathcal{G}_{k+2}(H)$. Since $f'(V) \in \mathcal{G}_{k+1}(H)$ (by assumption), we have $j \leq k$. Also, if $j < k$, then

$$f'(U_{j+1}) = \cdots = f'(U_{k+1}) = f'(V).$$

Therefore, $f'(V) \subset f'(U_j)$. On the other hand, $f'(V)$ is the orthogonal complement of $f'(U)$ and

$$\dim(f'(U) \cap f'(U_j)) = 1.$$ 

The subspaces $U$ and $U_j$ are connected by the sequence $U = U_0, U_1, \ldots, U_j$ with $j \leq k$ which implies that

$$\dim(f'(U) \cap f'(U_j)) \geq k + 2 - j \geq 2,$$

and we get a contradiction.
The case (2) is similar. We consider minimal \( t \) and maximal \( j \) such that \( f'(U_t) \) and \( f'(U_j) \) belong to \( G_{k+2}(H) \). Since \( f'(U) \) and \( f'(V) \) are elements of \( G_{k+1}(H) \), we have \( t \geq 1 \) and \( j \leq k \). As in the previous case, we have

\[
f'(U) = f'(U_0) = \cdots = f'(U_{t-1}) \quad \text{and} \quad f'(U_{j+1}) = \cdots = f'(U_{k+1}) = f'(V)
\]

if \( t > 1 \) and \( j < k \), respectively. Therefore,

\[
f'(U) \subset f'(U_t) \quad \text{and} \quad f'(V) \subset f'(U_j).
\]

Recall that \( f'(U), f'(V) \) are orthogonal \((k+1)\)-dimensional subspaces. This means that

\[
\dim(f'(U_t) \cap f'(U_j)) = 1 \text{ or } 2.
\]

On the other hand, \( U_t \) and \( U_j \) are connected by a sequence of \( j-t \) elements of \( G_{k+2}(H) \), where any two consecutive elements are adjacent and \( j-t \leq k-1 \); therefore,

\[
\dim(f'(U_t) \cap f'(U_j)) \geq k+2 - (j-t) \geq k+2 - (k-1) = 3
\]

which gives a contradiction again.

So, \( f'(U) \) and \( f'(V) \) both belong to \( G_{k+2}(H) \). Since for every \( U \in G_{k+2}(H) \) there is \( V \in G_{k+2}(H) \) such that \( U, V \) are compatible and \( \dim(U \cap V) = 1 \), the proof is complete. \( \square \)

3.3. The case \( i > 3 \)

In this case, we have \( m \geq i+1 \) by assumption. As in the previous subsection, we show that \( f \) is adjacency preserving.

Suppose that \( X, Y \in G_k(H) \) are adjacent. Then \( \dim(X + Y) = k + 1 \)

\[
\dim(X + Y) \perp = n - (k + 1) = mk + i - k - 1 = (m - 1)k + i - 1.
\]

Without loss of generality, we can assume that \( m = i + 1 \). In the case when \( m - i - 1 > 0 \), we choose mutually orthogonal \( k \)-dimensional subspaces

\[
X_1, \ldots, X_{m-i-1} \subset (X + Y) \perp
\]

(if \( m - i - 1 = 1 \), then we take any \( k \)-dimensional subspace \( X_1 \) orthogonal to \( X + Y \)), consider the subspaces

\[
M = (X_1 + \cdots + X_{m-i-1}) \perp \quad \text{and} \quad N = (f(X_1) + \cdots + f(X_{m-i-1})) \perp
\]

whose dimension is equal to \((i+1)k + i\) and observe that \( f \) sends any \( k \)-dimensional subspace of \( M \) to a \( k \)-dimensional subspace of \( N \).
Let \( m = i + 1 \). Two \( k \)-dimensional subspaces of \( H \) are adjacent if and only if their orthogonal complements are adjacent. In particular, we have
\[
\dim(X^\perp \cap Y^\perp) = n - k - 1 = (i + 1)k + i - k - 1 = ik + i - 1.
\]
Assume that \( f(X) \) and \( f(Y) \) are not adjacent. Then their orthogonal complements are also not adjacent and
\[
\dim(f(X)^\perp \cap f(Y)^\perp) < ik + i - 1.
\]
We set
\[
M_1 = X^\perp \cap Y^\perp \quad \text{and} \quad N_1 = f(X)^\perp \cap f(Y)^\perp.
\]
Then \( f \) sends any \( k \)-dimensional subspace of \( M_1 \) to a \( k \)-dimensional subspace of \( N_1 \), i.e. it induces an orthogonality preserving injection
\[
f_1 : \mathcal{G}_k(M_1) \to \mathcal{G}_k(N_1),
\]
where
\[
\dim N_1 < \dim M_1 = ik + i - 1.
\]
Note that \( f_1 \) is not adjacency preserving (otherwise, it is induced by a unitary or anti-unitary operator which contradicts the fact that \( \dim N_1 < \dim M_1 \)). Let us take any adjacent \( U, V \in \mathcal{G}_k(M_1) \) such that \( f(U), f(V) \) are not adjacent. Consider the subspace
\[
M_2 = U^\perp \cap V^\perp \cap M_1
\]
whose dimension is equal to
\[
\dim M_1 - k - 1 = (i - 1)k + i - 2.
\]
The map \( f \) sends any \( k \)-dimensional subspace of \( M_2 \) to a \( k \)-dimensional subspace contained in
\[
N_2 = f(U)^\perp \cap f(V)^\perp \cap N_1.
\]
Since \( \dim N_1 < \dim M_1 \) and \( f(U), f(V) \) are distinct non-adjacent \( k \)-dimensional subspaces of \( N_1 \), we have \( \dim N_2 < \dim M_2 \). As above, the restriction of \( f \) to \( \mathcal{G}_k(M_2) \) is not adjacency preserving.

Recursively, we establish that \( f \) induces a sequence of maps
\[
f_j : \mathcal{G}_k(M_j) \to \mathcal{G}_k(N_j), \quad j = 1, \ldots, i - 3,
\]
where
\[
\dim N_j < \dim M_j = (i - j + 1)k + i - j
\]
and every \( f_j \) is an orthogonality preserving injection, but it is not adjacency preserving. On the other hand,
\[
\dim M_{i-3} = 4k + 3
\]
and \( f_{j-3} \) can be considered as an orthogonality preserving injection of \( G_k(M_{i-3}) \) to \( G_k(M') \), where \( M' \) is a \((4k + 3)\)-dimensional complex Hilbert space containing \( N_{i-3} \). By the arguments from the previous subsection, \( f_{j-3} \) is adjacency preserving. We come to a contradiction which implies that \( f(X) \) and \( f(Y) \) are adjacent.

**Declaration of competing interest**

The author declared that he had no conflicts of interest with respect to their authorship or the publication of this article.

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